Applied Calculus

Spring 2008

Module 1: Functions

Introduction

This module introduces the notion of a **function** as well as many of the most important kinds of functions. We will meet linear, polynomial, exponential, logarithmic, and trigonometric functions, as well as many families of functions related to these. We will use these functions to model a large class of data as well as remind ourselves of their properties.

The primary goal of this module is to take a series of measurements from the real world–data–and translate them into a mathematical object–a model. Without being able to accomplish this, mathematics cannot be applied to anything in the real world. However, in the mathematics you have studied in the past, it is unlikely that much emphasis–if any–was placed on this process. Hence, this kind of mathematics may seem unfamiliar to you. By its very nature, Applied Mathematics cannot be learned by doing drill exercises from a textbook, or by using pencil and paper to solve problems; applied mathematicians are interested in being able to understand and perform direct computations that arise in applications, not canned problems used to illustrate ideas. For this reason, we will rely heavily on Mathematica. Mathematica is not to be considered a tool to learn the review material; rather, it is the way we will be doing mathematics. You are strongly encouraged to seriously approach learning Mathematica as a major goal for this course.

The data we will meet in this module will be ideal in the sense that it will exactly fit the mathematical models. Although this is not frequently the case in actual applications, it can be (for instance, in the case of interest, where the bank is using a function to compute the interest rate). Additionally, this will allow us to study the functions themselves without worrying about the degree to which they fit our data. In later modules, you will use these kinds of functions to model data that need not perfectly fit the model.

Many of the lectures in this module will be review of fundamental prerequisite material from the text (Review Lectures). It is assumed that all of the students have met this material before, so it will not be covered in depth. Other lectures (the Data Lectures) will begin with a specific set or sets of data from an application of the mathematics to be discussed. We will look at plots of the data and try to understand its behavior, and then see how to find the appropriate function to model the data. Using the model, we will see how to **extrapolate** and **interpolate** in order to predict the behavior of the function for a larger domain than that given. In particular, we will be able to study the **average rate of change** of each function over ranges of the independent variable and estimate the **instantaneous rate of change** of the function at given points.

In addition to the Data Lectures, there are two Supplemental Data Lectures that you should read. We will quickly see that the method we use to fit data to a model does not depend much on the kind of function we use. However, these lectures offer additional examples to supplement those we will cover in class.

For your final project, each group will be given a set of data and will determine a function to match the data. The group will then apply the techniques we have learned in order to study the behavior of the function.

The final week of the module (Monday, 10 September through Friday, 14 September) will be spent covering some additional prerequisite material for this class. There will be ample time for questions on the final projects.

Schedule for the Module

Date	Lecture	Assignment
Wed. 9^{nd} Jan	Introduction and Syllabus.	**Read:** Sections 0.1-2
Fri. 11^{th} Jan	Review Lecture 1: Functions; Linear, Polynomial, and Rational Functions	**Homework:** 0.1: 1, 7, 8*, 21, 35, 41, 44*, 53 0.2: 1, 5, 6*, 11, 17, 20*, 29 Supplemental Exercises 1 on page 5 below. **Read:** Sections 4.1-2, 4.4, 4.6 (through page 270)
Mon 14^{th} Jan	Review Lecture 2: Exponential and Logarithmic Functions	**Homework:** 4.1: 5, 15, 16*, 21, 24*, 27 4.2: 13, 16* 4.4: 21, 25, 30*, 45, 46* 4.6: 19, 20*, 27, 28* **Read:** Sections 8.1-2
Wed 16^{th} Jan	Review Lecture 3: Exponential and Logarithmic Functions	**Homework:** Mathematica Exercise, download from WebCT. **Read:** Sections 8.1-2
Fri 18^{th} Jan	Review Lecture 4: Trigonometric Functions	**Homework:** 8.1: 3, 7, 8* 8.2: 21, 24*, 29, 30*, 38*, 39, 40*
Wed 23^{th} Jan	Data Lecture 1: Projectile Motion	**Homework:** Supplemental Exercises 2 on page 5 below.
Fri 25^{th} Jan	Data Lecture 2: Sulfuric Acid pH	**Homework:** Prepare for the meeting with your group tutor.
Mon 28^{th} Jan	Data Lecture 3: Coffee Sales	**Homework:** Work on your projects **Read:** Sections 0.3
Wed 30^{th} Jan	Review Lecture 5: The Algebra of Functions (and final project questions)	**Homework:** 0.3: 3, 9, 10*, 15, 19, 22*, 23, 35, 36* **Read:** Section 0.4
Fri 1^{st} Feb	Review Lecture 6: Zeros of Functions (and final project questions)	**Homework:** 0.4: 3, 9, 10*, 19, 20*, 27, 39, 40* **Read:** Section 0.5
Mon 4^{th} Feb	Review Lecture 7: Exponents and Power Functions	**Homework:** 0.5: 15, 23, 24*, 47, 48*, 61, 87, 90* **Read:** Sections 1.1 and 1.2 (in preparation for Module 2).

Review Lecture 1: Functions; Linear, Polynomial, and Rational Functions

Supplemental Exercises 1

Suppose you are given the following data:

x	$f(x)$
1	-1
2	41
3	219
4	701
5	1727

These data points are points on the graph of a function $f(x)$ of the form

$$f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + F$$

where A, B, C, D, and F are constants. Determine values of these constants to fit the data points.

Perform all of your computations **by hand**. You may use a calculator for **simple arithmetic only**.

Review Lecture 3: Trigonometric Functions

Supplemental Exercises 2

1. Determine the amplitude, period, phase shift, and vertical shift of

   a. $f(x) = \cos(2x + 1) + 5$

   b. $f(x) = 7\sin(\pi - 3\pi x) - 3$

   c. $f(x) = \cos(x) + 1$

2. Determine a formula for a trigonometric function $f(x)$

   a. with period 7, amplitude 12, and vertical shift 9 that passes through the point $(3, 9)$.

   b. that repeats itself every 5π x-units, oscillates between a minimum of -1 and a maximum of 8, and reaches its maximum value at $x = 1$.

Data Lecture 1: Projectile Motion

A small, heavy ball bearing is thrown off of a bridge at a height of 50 meters with an initial velocity of 18 m/s at a 45° angle from the horizontal. The height of the object is recorded each second until the ball hits the ground. The height measurements are accurate to one tenth of a meter.

Time (seconds)	**Height (meters)**
0	50
1	54.1
2	48.4
3	32.9
4	7.6

Questions

- At what time does the object hit the ground?
- What is the vertical velocity of the object at $t = 0$ seconds? At $t = 1$ seconds? $t = 2$ seconds?
- What is the maximum height of the object? At what time is this height obtained?

Data Lecture 2: Sulfuric Acid pH

Several solutions of sulfuric acid (H_2SO_4) are mixed with different concentrations. The pH of each solution is measured with a pH meter and probe, and the results are given in the following table. In each case, the concentration is measured in **molarity, M**: moles of solute (sulfuric acid) per liters of solution (water). A **mole** is a unit of measurement that gives a specific number of molecules of a compound: a mole of sulfuric acid corresponds to 50 grams. The pH values are accurate to the nearest hundredth.

Concentration (M)	**pH**
0.00001	4.70
0.00005	4
0.0001	3.70
0.0006	2.92
0.0008	2.80
0.001	2.70

Questions

- How does the pH appear to change as the concentration changes?
- What are the units of the rate of change? What does the rate of change tell you?
- How does the rate of change appear to behave? Does a slight increase in concentration make more or less of a difference at large concentrations compared to small concentrations?

Data Lecture 3: Coffee Sales

The owner of a coffee shop is well aware of the fact that the sales of her hot beverages vary with the seasons. In an attempt the model this variation, she has collected the following data of daily sales figures. She chose every 40th day as a sufficient representative, but also included day 357, the day on which the sales are maximal, and a few other useful days. In the following data, the days are measured from the beginning of the calender year (so that January 1st, 2004 corresponds to $t = 0$).

Day (days)	**Sales ($)**
0	355.29
40	329.13
80	276.48
84	271.00
120	221.33
160	188.82
200	193.75
240	233.89
280	290.94
320	338.90
357	356.00
360	355.92
400	334.24
440	283.75

Questions

- Describe the behavior of the sales as a function of time. What trends do you expect?
- On what day do you expect the minimum sales?
- Describe the behavior of the rate of change of sales.
- On what day are the sales figures changing the fastest?

Supplemental Data Lecture 1: Compound Interest

A student has two bank accounts. The first is compounded annually, and the second is compounded continuously. The student deposited $100 into each account. The tables below shows the balance of each account every year for five years. Each value is rounded to the nearest cent

Compounded Annually

Time (years)	**Balance (dollars)**
0	100
1	105.40
2	111.09
3	117.09
4	123.41
5	130.08

Compounded Continuously

Time (years)	**Balance (dollars)**
0	100
1	105.55
2	111.40
3	117.59
4	124.11
5	131.00

Refer to Mathematica notebook M1_ L4_ bankaccounts.nb.
Questions

- Compare the rates at which the balances of the two accounts are increasing.
- For each of the two accounts, how is the rate of increase changing?
- Is there a significant difference between the growth of the two accounts? Does the difference become more or less significant with time?
- What is the value of each account after 20 years? 30 years?

Supplemental Data Lecture 2: Ferris Wheel Height

Students are trying to model the height of a car in a Ferris wheel. The students take several measurements of the height of a car in the Ferris wheel that starts at a (minimum) height of 5.4 m at time 0. The students can only measure the height of the car when it less than 15 meters above the ground. The measurements they are able to take are in the table below. They are accurate to the nearest hundredth of a meter.

Time (min:sec)	**Height (m)**
0:00	5.40
0:12	5.71
0:16	5.97
0:25	6.80
0:40	8.99
11:40	8.74
11:48	7.50
12:00	6.17
12:18	5.40
12:30	5.71
12:47	7.28

Refer to Mathematica notebook `M1_` *L7_ ferriswheel.nb.*
Questions

- What is the period of the function?
- What is the phase shift?
- How much information is required in order to determine the amplitude?