

Module 4:
Areas under curves and Anti -derivatives

Problem 1: Velocity from Acceleration.

Problem 2: Mass from Concentration, in the
outlet pipe

Problem 3: Mass from Concentration, in the
settlement tank

In this module we will see how to calculate total accumulations of quantities from the rate of accumulation. This includes calculating total amounts from densities and concentrations. This type of calculation can also be used to find the area with a curved boundary. The formal mathematical techniques we develop here will be studied in Module 5.

Schedule

Date	Lecture	Assignment
Wed 29 th Oct	Problem 1: Speed from acceleration	None
Fri 31 th Oct	Finding anti-derivatives	Homework 1: Sec 6.1: 1, 2*, 3, 4*, 13, 16*, 37, 40* Sec 8.3: 35, 36*, 39, 40*
Fri 7 th Nov	Areas under curves	Homework 2: Exercise on page 9
Mon 10 th Nov	Problem 2: Finding total mass flowing into the settlement tank	Homework 3: Exercise on page 14
Wed 12 th Nov	Problem 3: Finding total mass in a settlement tank	Homework 4: Exercise on page 19
Mon 24 th Nov	Present posters of Projects in MSC	

1 Problem 1

1.1 Statement

The most critical stage of a 100m race is as the athlete accelerates out of the blocks. The athlete will be able to maintain the speed they manage to reach in this phase. Below are the values of an athlete's acceleration in the first 0.8 seconds of a practice run. What is the athlete's speed 0.8 seconds into the run?

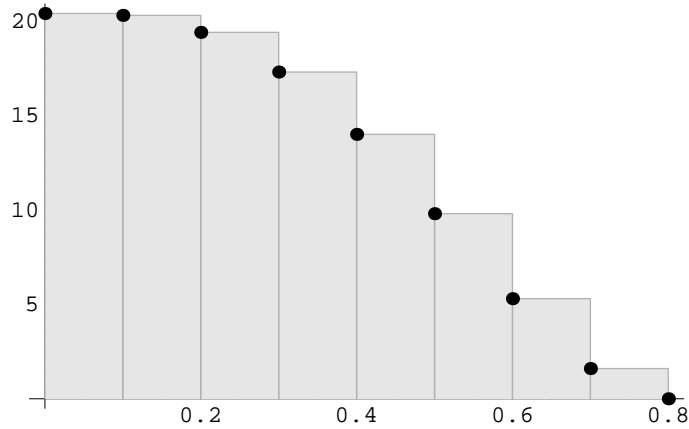
Time (sec)	Acceleration (m/s ²)
0	20.4
0.1	20.3
0.2	19.4
0.3	17.3
0.4	14.0
0.5	9.8
0.6	5.3
0.7	1.6
0.8	0

1.2 Estimation from the Data

If we assume that the athlete maintains each of these accelerations for the full interval, then we could calculate their speed as follows:

Acceleration	Length of interval	Increase in speed
20.4	0.1	$20.4 \times 0.1 = 2.04$
20.3	0.1	$20.3 \times 0.1 = 2.03$
19.4	0.1	$19.4 \times 0.1 = 1.94$
17.3	0.1	$17.3 \times 0.1 = 1.73$
14.0	0.1	$14.0 \times 0.1 = 1.40$
9.8	0.1	$9.8 \times 0.1 = 0.98$
5.3	0.1	$5.3 \times 0.1 = 0.53$
1.6	0.1	$1.6 \times 0.1 = 0.16$
		$\sum_{i=0}^8 a_i \times 0.1 = 10.81$

Each of these multiplications can be represented as an area of a rectangle and together can be draw as:



Total area is 10.81.

1.3 Improved approximations using a fitted model

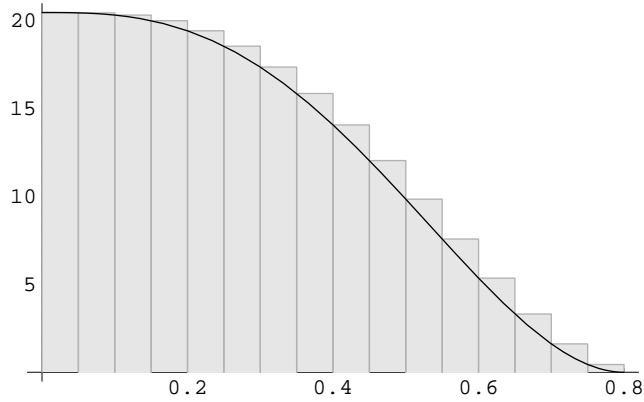
The athlete does not, however, maintain their acceleration over the whole 0.1 second interval, their acceleration is continually decreasing. If we had their acceleration every 0.05 seconds and repeated the calculation above we would obtain a more accurate approximation. To find the athlete's acceleration at these intermediate times we first fit an interpolation function to the data. If we find the best fitting model of the form $y = at^4 + bt^3 + c$ using the techniques of Module 3 we would find that this data is approximated by with the function

$$a(t) = 150.51t^4 - 160.28t^3 + 20.42$$

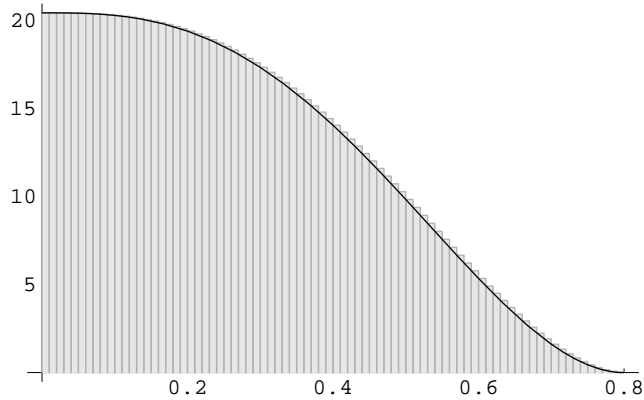
So the speed would be $\sum_{i=0}^{15} a(0.05 \times i) \times 0.05 = 10.299$. This can be calculated in Mathematica with the commands:

```
acc[t_]=150.51 t^4-160.28t^3+20.42
Sum[acc[0.05*i]*0.05,{i,0,0.8/0.05-1}]
```

As before, we can represent this as the sum of the areas of rectangles (each rectangle has a height of $a(0.05 \times i)$ and a width 0.05. These are pictured below:



It would be even more accurate if only assumed constant acceleration for 0.01 second time intervals, that is, calculate $\sum_{i=0}^{79} a(0.01 \times i) \times 0.01$. The Mathematica command is `Sum[acc[0.01*i]*0.01,{i,0,0.8/0.01-1}]` and the result is 9.89. The illustration of this is



Here the bars are so narrow that sum in approximately equal to the area under the curve.

1.4 Best possible approximation

The shorter the time interval over which we assume that the acceleration is constant the more accurate the approximation is. So if we take find the limit of these approximations as the interval width tended to zero we would have best possible approximation of the athlete's speed, and the *exact* value of the area under the curve. That is $\lim_{dt \rightarrow 0} \sum_{i=0}^{0.8/dt-1} a(dt \times i) \times dt$ in Mathematica this is calculated with `Limit[Sum[acc[dt*i]*dt,{i,0,0.8/dt-1}],dt->0]`, and the

answer we get is 9.78853. So the athlete's speed, to 3 significant figures, at the end of this spurt of acceleration is 9.79 m/s.

1.5 Another method

Acceleration is, by definition, the instantaneous rate of change of velocity (speed). So if $a(t)$ represents the acceleration of a body at time t and $v(t)$ represents the speed of that body at time t then

$$\frac{dv}{dt} = a(t)$$

In this problem the 'body' is the athlete, we know their acceleration ($a(t) = 149.54t^4 - 159.51t^3 + 20.42$) and wish to know the athlete's speed at $t = 0.8$. In other words if we differentiated the function we want to know about (speed) we would get $149.54t^4 - 159.51t^3 + 20.42$ (acceleration), so we have to 'undo' the differentiation that led to $149.54t^4 - 159.51t^3 + 20.42$. Another way to say this, is that to find the speed we need to solve the differential equation

$$\frac{dv}{dt} = 150.51t^4 - 160.28t^3 + 20.42$$

This equation is also called the model of the problem. Mathematica will solve this differential equation in response to the command

```
DSolve[v'[t]==150.51 t^4-160.28t^3+20.42,v[t],t]
```

The solution it gives is $20.42t - 40.07t^4 + 30.1t^5 + C_1$, where C_1 denotes the arbitrary constant. We do not have to worry about the value of this constant if we think of our problem as finding the increase in speed from $t = 0$ to $t = 0.8$ then we are finding $v(0.8) - v(0)$ and this is

$$(20.42 \times 0.8 - 40.07 \times 0.8^4 + 30.1 \times 0.8^5 + C_1) - (0 + C_1)$$

and the value of this is 9.79, to 3 significant figures.

2 Anti-derivatives and Areas

2.1 Summary of anti-derivatives

The table below summarizes the anti-derivatives that should be memorized:

$f(x)$	$\int f(x)dx$
x^n , for $n \neq -1$	$\frac{1}{n+1}x^{n+1} + C$
$\frac{1}{x}$	$\ln(x) + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\cos(ax)$	$\frac{1}{a}\sin(ax) + C$
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + C$

For any other $f(x)$ use the Mathematica command `DSolve[F'[x]==f[x],F[x],x]`

2.2 Calculating areas under curves

2.2.1 Theoretical summary

We have seen that we can find the area under a curve inbetween two values of x we can sum the areas of rectangles under the curve, and take the limit as the width of the rectangles tends to zero. In other words calculate:

$$\lim_{dx \rightarrow 0} \sum_{i=0}^{\lfloor (b-a)/dx \rfloor} f(a + idx) dx$$

In Mathematica the command for this is

```
Limit[ Sum[ f[a+i*dx], {i,0,(b-a)/dx-1} ], dx->0]
```

2.2.2 Example

For the function $f(x) = -2x^3 + 8x$, calculate the following areas:

- The sum of the areas of rectangles under the curve of width 0.2 from $x = 0.6$ to $x = 1.6$. Provide a graphical illustration.
- The sum of the areas of rectangles under the curve of width 0.05 from $x = 0.6$ to $x = 1.6$. Provide a graphical illustration.
- The sum of the areas of rectangles under the curve of width 0.01 from $x = 0.6$ to $x = 1.6$. Provide a graphical illustration.
- The area under the curve from $x = 0.6$ to $x = 1.6$.

- e. The sum of the areas of rectangles under the curve of width 0.2 from $x = 0.6$ to $x = 3.6$. Provide a graphical illustration.
- f. The sum of the areas of rectangles under the curve of width 0.01 from $x = 0.6$ to $x = 3.6$. Provide a graphical illustration.
- g. The area under the curve from $x = 0.6$ to $x = 3.6$. (**Note** Areas below the x -axis are negative).

See file `ares.nb` for the solution.

2.2.3 Exercise for Homework 2

For the function $f(x) = \sin(x)$, calculate the following:

- a. The sum of the areas of rectangles under the curve of width 0.2 from $x = 0$ to $x = \frac{\pi}{2}$. Provide a graphical illustration.
- b. The sum of the areas of rectangles under the curve of width 0.05 from $x = 0$ to $x = \frac{\pi}{2}$. Provide a graphical illustration.
- c. The sum of the areas of rectangles under the curve of width 0.01 from $x = 0$ to $x = \frac{\pi}{2}$. Provide a graphical illustration.
- d. The area under the curve from $x = 0$ to $x = \frac{\pi}{2}$.
- e. The sum of the areas of rectangles under the curve of width 0.2 from $\frac{\pi}{6}$ to $\frac{2\pi}{3}$. Provide a graphical illustration.
- f. The sum of the areas of rectangles under the curve of width 0.01 from $\frac{\pi}{6}$ to $\frac{2\pi}{3}$. Provide a graphical illustration.
- g. The area under the curve from $\frac{\pi}{6}$ to $\frac{2\pi}{3}$.

Copy your solutions into a word document and submit within WebCT.

3 Problem 2

3.1 Statement

A factory needs to use a certain toxic chemical in its processing. There is a steady flow, of 500 litres of water per hour, through the machinery that washes the chemical into a settlement tank. The factory runs its process for eleven hours and then stops for one hour to allow the equipment to be washed clear of the chemical. The table below shows the concentration of chemical in the outlet pipe, at different times during a typical twelve hour shift. Our task is to calculate the total mass of chemical in the tank when it is full. The settlement tanks measure 18m by 5m and are 2m deep.

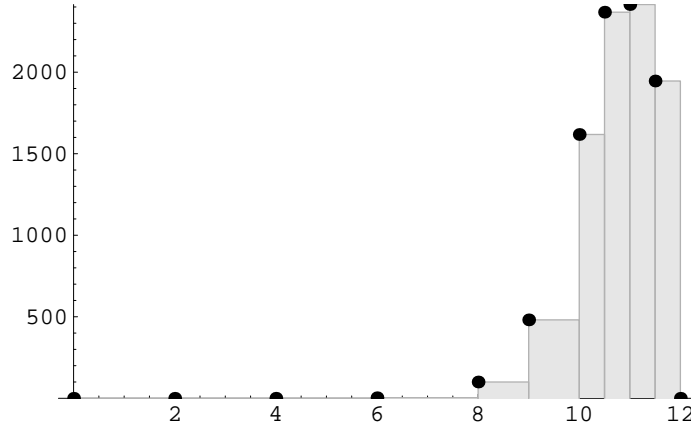
Time (Hours)	Concentration (mg/L)
0	0.00
2	0.00
4	0.28
6	3.33
8	100
9	481
10	1618
10.5	2368
11	2415
11.5	1946
12	0.00

3.2 Estimation from the Data

If we assume that the concentration is constant over the time intervals

Concentration	Length of interval	Amount of chemical
0.00	4.0	$0.00 \times 4.0 \times 500 = 0.00$
0.28	2.0	$0.28 \times 2.0 \times 500 = 0.56$
3.33	2.0	$3.33 \times 2.0 \times 500 = 6.66$
100	1.0	$100 \times 1.0 \times 500 = 100$
481	1.0	$481 \times 1.0 \times 500 = 481$
1618	0.5	$1618 \times 0.5 \times 500 = 809$
2368	0.5	$2368 \times 0.5 \times 500 = 1184$
2415	0.5	$2415 \times 0.5 \times 500 = 1207.5$
1946	0.5	$1946 \times 0.5 \times 500 = 973$
		$\sum_{i=0}^9 a_i \times dt_i \times 500 = 2\,380\,860$

Each of these multiplications can be represented as an area of a rectangle times 500, and together can be draw as:



Total area is 4761.72. So the total amount of chemical flowing into the tank in one shift is 4761.72×500 , i.e., 2 380 860.

3.3 Improved approximations using a fitted model

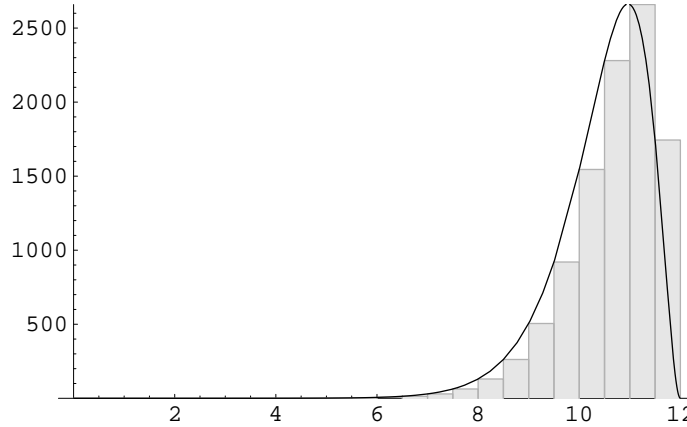
The concentration does not, however, remain constant for the whole time interval, it changes continuously within the intervals. If we had a measurement of the concentration every 30 minutes we could assume that it remained constant over these intervals, and obtain a more accurate estimate. The closest we can get to 30 minute measurements is to fit an interpolation function to the data. In Module 3 we saw a very similar problem and using that process we see that the concentration, c , at time t may be written as:

$$c(t) = 18302(12 - t)^2 e^{1.9291(12-t)}$$

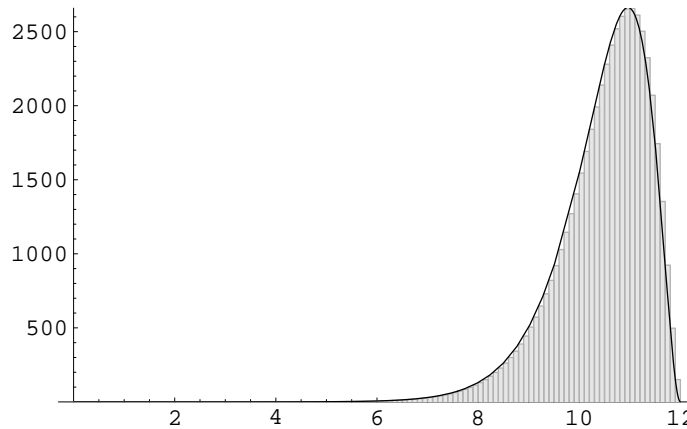
So the '30 minute' estimate would be $500 \sum_{i=0}^{23} c(0.5i) 0.5 = 2\,540\,896$. To calculate this in Mathematica we can type:

```
con[t_]=18302*(12-t)^2*E^(1.9291(12-t));
500*Sum[con[0.5*i]*0.5,{i,0,23}]
```

The graphical representaion of this sum is



It would be more accurate still to assume that the concentration was constant for only 6 minutes, and calculate $500 \sum_{i=0}^{119} 0.1c(0.1i)$. The Mathematica command that does this is `500*Sum[con[0.1*i]*0.1,{i,0,12/0.1-1}]` and the result is 2549429. The illustration of this is



3.4 Best possible approximation

Taking this process to its limit, the best approximation we could make by this process is

$$\lim_{dt \rightarrow 0} 500 \sum_{i=0}^{12/dt-1} c(i dt) dt$$

The Mathematica command that would calculate this is

$$\text{Limit}[500*\text{Sum}[\text{con}[i*dt]*dt,\{i,0,12/dt-1\}],dt->0]$$

Unfortunately Mathematica can have problems calculating limits for transcendental functions, so the best we can do is to set dt to a very small value, e.g., 10^{-10} and calculate the sum. The answer is 2.54944×10^6 . This is in mg so is equal to 2.54944kg.

3.5 Differential Model

If the amount of a chemical is evenly distributed through out a volume of water then the concentration is given by the word formula

$$\text{Concentration} = \frac{\text{Amount}}{\text{Volume}}$$

In this problem, however, the amount in the water changes continuously. So if we focus on a very short time interval, from t to $t + dt$ say, and think about the water that leaves the outlet pipe in this time interval, we can say:

$$\begin{aligned} \text{Average Concentration} \\ \text{of water that passes} \\ \text{between time } t \text{ and time } t + dt &= \frac{\text{Amount of Chemical to pass in } dt \text{ hours}}{\text{Volume of water to pass in } dt \text{ hours}} \\ &= \frac{A(t + dt) - A(t)}{500 dt} \end{aligned}$$

where we are using $A(t)$ to represent the total mass of chemical to have passed through the pipe in t hours. The instantaneous concentration at time t will, therefore be given by:

$$c(t) = \lim_{dt \rightarrow 0} \frac{A(t + dt) - A(t)}{500 dt} = \frac{1}{500} \frac{dA}{dt}$$

This allows to say that the differential model of the problem is:

$$\frac{dA}{dt} = 500c(t) = 500 \left(18302(12 - t)^2 e^{1.9291(12-t)} \right)$$

To solve this equation in Mathematica we type:

```
DSolve[A'[t]==500*(bestA*(12-t)^2*E^(bestB*(12-t))),A[t],t]
```

The solution that Mathematica gives is

```
t 1.9290836612579962
C[1] + E
      2
*(t -0.010500525421595833 + t  0.00041940429675416064 + 0.06583748987344133)
```

If we rephrase the main question to be find the increase in the amount to flow into the tank in a 12 hour shift, then we are asking for $A(12) - A(0)$ and so will not have to worry about the value of the arbitrary constant $C[1]$. In Mathematica we type $A[12]-A[0]$ and we get 2.54944×10^6 .

3.6 Final calculation

The solution of 2.54944×10^6 mg is the mass of chemical to flow into the tank in one shift. To calculate the total amount to flow into a full tank we must find out how many shifts it takes to fill a tank. Volume of a tank is $18 \times 5 \times 2 = 180\text{m}^3$. The total amount of water in a full tank is, therefore 180 000L (a volume of 1m^3 holds 1 000L of water). The flow rate is 500L per hour so it takes $\frac{180\,000}{500} = 360$ hours to fill the tank. There are 12 hours in a shift, so it takes 30 shifts to fill a tank. Hence the total mass of chemical to flow into the tank is $30 \times 2.54944 \times 10^6 = 7.64833 \times 10^7$ mg or 76.4833kg.

3.7 Exercise for Homework 3

1. Another factory runs on on 4 hour shifts and the concentration function is $c(t) = 3.2t(4 - t)e^{-2.5(4-t)t}$. The flow rate is 100 L/hour.
 - (a) What is the total amount of chemical to flow into the tank in the first 2 hours of the shift?
 - (b) What is the total amount of chemical to flow into the tank in a shift?
 - (c) A full tank holds 1600L. What is the mass of the chemical in a full tank?
2. At another factory the concentration function, for a 6 hour shift, is know to be $c(t) = 3(6 - t)e^{-0.01(6-t)t}$. The flow rate is 200 L/hour.
 - (a) State the differential model for $A(t)$, the total amount of chemical to flow out in the first t hours of the shift.
 - (b) Find the increase in the amount of chemical to flow into the tank between the second and fifth hour of the shift.

Copy your solutions into a word document and submit within Moodle.

4 Problem 3

4.1 Statement

Ten days after a settlement tank is full, concentration measurements are taken at various depths in the tank. The results are below. Our task is to estimate the total amount of chemical in the tank. Remember that the tank measures 18m by 5m and is 2m deep.

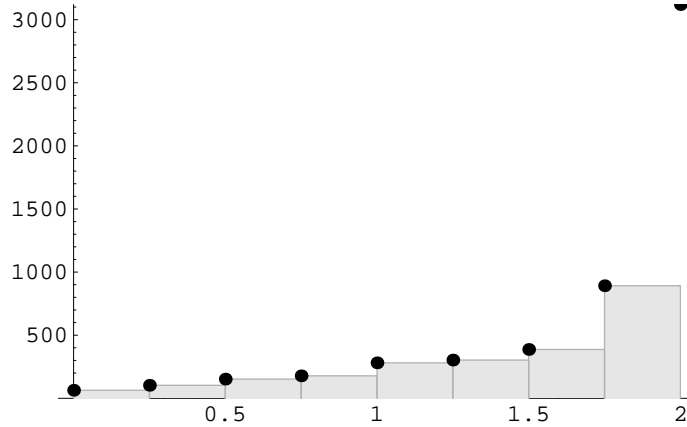
Depth (m)	Concentration (mg/L)
0.0	63.5
0.25	103
0.5	152
0.75	178
1.0	281
1.25	303
1.5	387
1.75	892
2.0	3120

4.2 Estimation from the Data

If we assume that the concentration is constant for the 25cm intervals between measurements

Concentration	Length of interval	Amount of chemical
63.5	0.25	$63.5 \times 0.25 \times 5 \times 18 \times 1000 = 1.43 \times 10^6$
103	0.25	$103 \times 0.25 \times 5 \times 18 \times 1000 = 2.32 \times 10^6$
152	0.25	$152 \times 0.25 \times 5 \times 18 \times 1000 = 3.42 \times 10^6$
178	0.25	$178 \times 0.25 \times 5 \times 18 \times 1000 = 4.01 \times 10^6$
281	0.25	$281 \times 0.25 \times 5 \times 18 \times 1000 = 6.32 \times 10^6$
303	0.25	$303 \times 0.25 \times 5 \times 18 \times 1000 = 6.82 \times 10^6$
387	0.25	$387 \times 0.25 \times 5 \times 18 \times 1000 = 8.71 \times 10^6$
892	0.25	$892 \times 0.25 \times 5 \times 18 \times 1000 = 20.1 \times 10^6$
		$\sum_{i=0}^9 a_i \times dh_i \times 5 \times 18 \times 1000 = 53\,088\,800$

Each of these multiplications can be represented as an area of a rectangle times $18 \times 5 \times 1000$, and together can be drawn as:



Total area is 589.875. So the total amount is $589.875 \times 18 \times 5 \times 1000$, i.e., 53 088 800.

4.3 Improved approximations using a fitted model

The concentration does not, however, remain constant for the whole 25cm, it changes continuously within the depths. If we had a measurement of the concentration every 10cm we could assume that it remained constant over these intervals, and obtain a more accurate estimate. The closest we can get to 10cm measurements is to fit an interpolation function to the data. Tutorialal groups from previous semesters have fitted a model of the form

$$c(h) = \frac{a}{\sqrt{b} - \sqrt{h}}$$

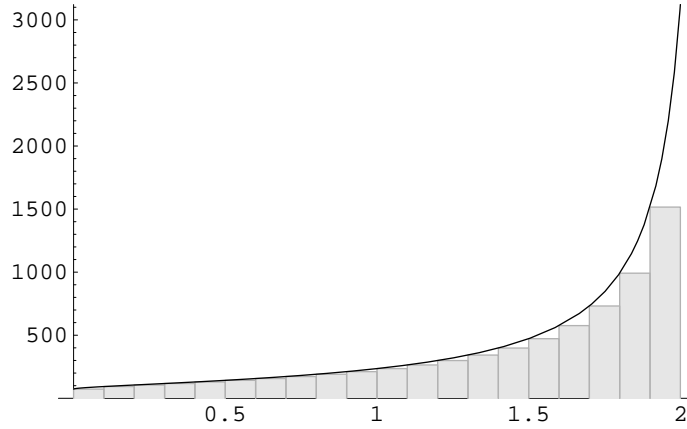
to this data and found that the optimal model was:

$$c(h) = \frac{105.5}{1.45 - \sqrt{h}}$$

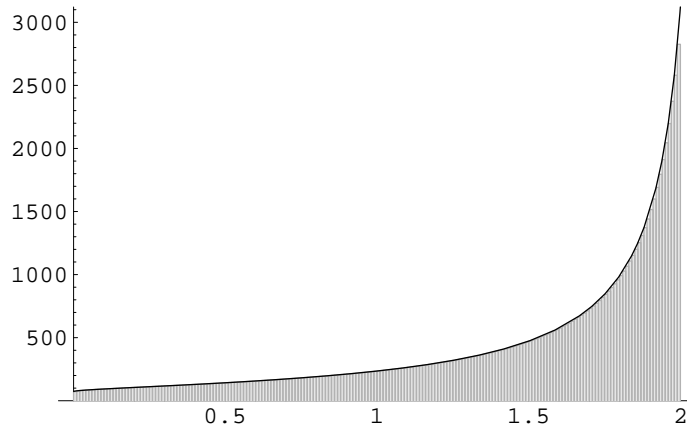
So the '10cm' estimate would be $90\,000 \sum_{i=0}^{19} c(0.1i) 0.1 = 64\,328\,200$. To calculate this in Mathematica we can type:

```
con[h_]=105.5138/(1.44801-Sqrt[h]);
90000*Sum[con[0.1*i]*0.1,{i,0,19}]
```

The graphical representaion of this sum is



It would be more accurate still to assume that the concentration was constant for only 1cm, and calculate $90\,000 \sum_{i=0}^{199} 0.01c(0.01i)$. The Mathematica command that does this is `90000*Sum[con[0.01*i]*0.01,{i,0,2/0.01-1}]` and the result is 75 127 900. The illustration of this is



4.4 Best possible approximation

Taking this process to its limit, the best approximation we could make by this process is

$$\lim_{dt \rightarrow 0} 90\,000 \sum_{i=0}^{2/dh-1} c(i\,dh)\,dh$$

The Mathematica command that would calculate this is

```
Limit[90000*Sum[con[i*dh]*dh,{i,0,2/dh-1}],dh->0]
```

Unfortunately Mathematica can have problems calculating limits for transcendental functions, so the best we can do is to set dh to a very small value. A further

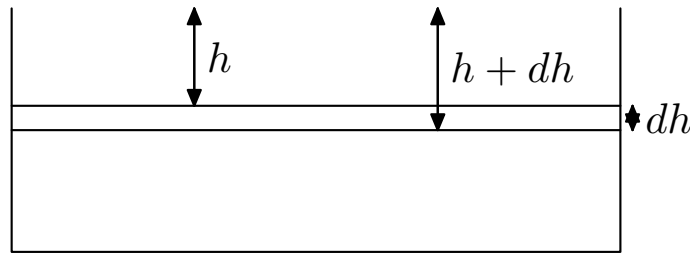
problem arises because Mathematica refuses to calculate the sum for values of dh below 0.00001. So the best approximation we can calculate is 7.64722×10^7 . This is in mg so is equal to 76.4722kg.

4.5 Differential Model

If the amount of a chemical is evenly distributed through out a volume of water then the concentration is given by the word formula

$$\text{Average Concentration} = \frac{\text{Amount}}{\text{Volume}}$$

In this problem, however, the amount in the water changes continuously as we go deeper down in the tank. So if we focus on the thin 'sheet' of water illustrated below



between the depths of h and $h + dh$, we can say:

$$\begin{aligned} \text{Average Concentration} \\ \text{between depths of} \\ h \text{ and } h + dh \end{aligned} &= \frac{\text{Amount of Chemical within the 'sheet'}}{\text{Volume of water with in the 'sheet'}} \\ &= \frac{A(h + dh) - A(h)}{18 \times 5 \times 1000 \times dh} \end{aligned}$$

where we are using $A(h)$ to represent the total mass of chemical that is contained in the top hm of the tank. The instantaneous concentration at depth h will, therefore be given by:

$$c(h) = \lim_{dh \rightarrow 0} \frac{A(h + dh) - A(h)}{90000 dh} = \frac{1}{90000} \frac{dA}{dh}$$

This allows to say that the differential model of the problem is:

$$\frac{dA}{dh} = 90000c(h) = \frac{90000a}{\sqrt{b} - \sqrt{h}}$$

To solve this equation in Mathematica we type:

```
DSolve[A'[h]==90000*bestA/(Sqrt[bestB]-Sqrt[h]),A[h],h]
```

The solution that Mathematica gives is

```
C[1] + 9.496244760263806e6  
*(Sqrt[h] -2. + -2.8960278685557594 Log[-1.55876932e8 + Sqrt[h] 1.07648779e8])
```

If we rephrase the main question to be find the increase in the amount from the top of the tank to the bottom, then we are asking for $A(2) - A(0)$, and we will not have to worry about the value of the arbitrary constant $C[1]$. In Mathematica we type $A[2]-A[0]$ and we get 7.64764×10^7 .

4.6 Final calculation

This amount is similar, but not equal to, the amount that we estimated as flowing into the tank. The difference in the amounts is $7.64833 \times 10^7 - 7.64764 \times 10^7 = 6921.97\text{mg}$. This is the mass of chemical that we estimate has been lost from the tank during the time that the water has been settling. This may, or may not, be a large amount, it depends on the nature of the chemical.

4.7 Exercise for Homework 4

1. At a different time the concentration function, for concentrations at different depths in a settlement tank is $c(h) = \frac{120}{\sqrt[3]{2.05} - \sqrt[3]{h}}$.
 - (a) Estimate the mass of chemical in the tank by assuming that the concentration is constant for 'sheets' of depth 0.25cm.
 - (b) Find the best estimate that Mathematica will allow, for the mass of chemical in the tank.
2. Another tank measures 10m by 4m and is 1.5m deep. The concentration function is found to be $c(h) = \frac{100}{\sqrt{1.6} - \sqrt{h}}$.
 - (a) State a differential model for $A(h)$, the mass of chemical in the top h metres of the tank.
 - (b) Find the increase in the mass of chemical from the top to the bottom of the tank.

Copy your solutions into a word document and submit within Moodle.