Computational methods in the study of symplectic quotients, Lecture 3

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Minicourse Abstract

Let G be a compact Lie group and let V be a unitary G-representation. Then there is a quadratic moment map $J : V \to \mathfrak{g}^*$ with respect to which V is a Hamiltonian manifold. Letting Z denote the zero fiber $J^{-1}(0)$ of the moment map, the corresponding *symplectic quotient* is given by $M_0 = Z/G$. It has the structure of a symplectic stratified space as well as a semialgebraic set, and it is equipped with an *algebra of regular functions* $\mathbb{R}[M_0]$, a Poisson subalgebra of its algebra of smooth functions.

In these lectures, we will introduce methods of computing the algebra $\mathbb{R}[M_0]$ of regular functions on such a symplectic quotient using methods from invariant theory and computational algebraic geometry. In addition, we will explain how these computations can be used to observe and verify properties of the symplectic quotient. Topics will include using Groebner bases to compute invariant polynomials, elimination theory, and methods of computing Hilbert series of Cohen-Macaulay algebras. In addition, we will introduce the software packages *Mathematica* and *Macaulay2* for these kinds of computations.

Minicourse Lectures

1. Invariant theory and Gröbner bases
2. Singular symplectic reduction and regular functions on symplectic quotients
3. **The Hilbert series of the regular functions on a symplectic quotient**
4. Elimination theory and the nonabelian case

Lecture 3: The Hilbert series of the regular functions on a symplectic quotient

1. Hilbert Series of Graded Algebras
2. Molien's Formula for Finite Groups
3. Laurent Coefficients for Invariants of a Finite Group
4. The Hilbert Series of (Symplectic) Circle Quotients
5. An Algorithm for the Hilbert Series of Symplectic Circle Quotients

Hilbert Series of Graded Algebras

Graded algebras

The algebra $\mathbb{K}[x_1, \dots, x_n]$ is a **graded algebra** by defining

$$\mathbb{K}[x_1, \dots, x_n]_j = \{\text{homogeneous polynomials of degree } j\}, \quad j = 0, 1, 2 \dots.$$

Then

- each $\mathbb{K}[x_1, \dots, x_n]_j$ is a vector subspace of $\mathbb{K}[x_1, \dots, x_n]$,
- $\mathbb{K}[x_1, \dots, x_n]_j \cap \mathbb{K}[x_1, \dots, x_n]_k = \{0\}$ for $j \neq k$,
- every element of $\mathbb{K}[x_1, \dots, x_n]$ is a finite sum of elements of the $\mathbb{K}[x_1, \dots, x_n]_j$, and
- the product of a nonzero element of $\mathbb{K}[x_1, \dots, x_n]_j$ with a nonzero element of $\mathbb{K}[x_1, \dots, x_n]_k$ is an element of $\mathbb{K}[x_1, \dots, x_n]_{j+k}$.

Not that each $\mathbb{K}[x_1, \dots, x_n]_j$ is finite-dimensional.

The Hilbert Series of a Graded Algebra

Definition

Given a graded algebra $R = \oplus_{j=0}^{\infty} R_j$, the **Hilbert series of** R is the formal power series

$$\mathrm{Hilb}_R(t) = \sum_{t=0}^{\infty} \dim(R_j) t^j.$$

The Hilbert Series of a Graded Algebra, Example

For $R = \mathbb{K}[x]$, each $\mathbb{K}[x]_j$ has dimension 1 (spanned by x^j).

The Hilbert series is

$$\mathrm{Hilb}_{\mathbb{K}[x]}(t) = 1 + t + t^2 + t^3 + \cdots = \sum_{j=0}^{\infty} t^j.$$

This is the power series of

$$\frac{1}{1-t}.$$

The Hilbert Series of a Graded Algebra, Example

For $R = \mathbb{K}[x_1, \ldots, x_n]$, each $\mathbb{K}[x_1, \ldots, x_n]_j$ has dimension $\binom{n+j-1}{n-1}$.

The Hilbert series is

$$\mathrm{Hilb}_{\mathbb{K}[x]}(t) = \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} t^j.$$

This is the power series of

$$\frac{1}{(1-t)^n}.$$

The Hilbert Series of a Graded Algebra

Theorem (Hilbert)

If $R = \oplus_{j=0}^{\infty} R_j$ *is a finitely generated graded algebra over* $\mathbb{K} = R_0$*, then* $\mathrm{Hilb}_R(t)$ *is the power series of a rational function with radius of convergence at least* 1*.*

See Derksen–Kemper [1, Proposition 1.4.5].

The Hilbert Series of a Graded Algebra, Example

Example

If R is a subalgebra of $\mathbb{K}[x_1, \dots, x_n]$ generated by a single homogeneous polynomial f of degree d, then

$$\mathrm{Hilb}_R(t) = 1 + t^d + t^{2d} + t^{3d} + \cdots$$

This is the power series of

$$\frac{1}{1 - t^d}.$$

Example

If R is a subalgebra of $\mathbb{K}[x_1, \dots, x_n]$ generated by r *algebraically independent* homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$, then

$$\mathrm{Hilb}_R(t) = \frac{1}{(1 - t^{d_1})(1 - t^{d_2}) \cdots (1 - t^{d_r})}.$$

The Hilbert Series of a Graded Algebra, Example

In $\mathbb{C}[x_1, x_2]$, let $R = \mathbb{C}[x_1x_2, x_1^2, x_2^2]$.

The generators $\{x_1x_2, x_1^2, x_2^2\}$ are not algebraically independent.

The relations are

$$(x_1x_2)^2 - x_1^2x_2^2.$$

The Hilbert series is

$$\mathrm{Hilb}_R(t) = \frac{1+t^2}{(1-t^2)^2} = \frac{1-t^4}{(1-t^2)^3}.$$

The Laurent Expansion of the Hilbert Series

In the examples considered above, the Hilbert series $\mathrm{Hilb}_R(t)$ has a pole at $t = 1$ (this is *usually* the case).

However, we can expand $\mathrm{Hilb}_R(t)$ as a **Laurent series**

$$\mathrm{Hilb}_R(t) = \sum_{j=0}^{\infty} \gamma_j (1-t)^{j-D} = \frac{\gamma_0}{(1-t)^D} + \frac{\gamma_1}{(1-t)^{D-1}} + \frac{\gamma_2}{(1-t)^{D-1}} + \cdots$$

We call the γ_j the **Laurent coefficients** of the Hilbert series.

The number D is the **Krull dimension** of R, the maximum number of algebraically independent elements of R, denoted $\dim(R)$.

When $\mathbb{K} = \mathbb{C}$, it corresponds to the dimension of the variety.

The Laurent Expansion of the Hilbert Series, Examples

Example

For $R = \mathbb{K}[x]$,

$$\mathrm{Hilb}_R(t) = \frac{1}{1-t}$$

is already expanded at $t = 1$.
We have $\dim(R) = 1$, $\gamma_0 = 1$ and $\gamma_j = 0$ for $j > 0$.

Example

For $R = \mathbb{K}[x_1, \dots, x_n]$,

$$\mathrm{Hilb}_R(t) = \frac{1}{(1-t)^n}$$

is already expanded at $t = 1$.
We have $\dim(R) = n$, $\gamma_0 = 1$ and $\gamma_j = 0$ for $j > 0$.

The Laurent Expansion of the Hilbert Series, Examples

When R is generated by a single polynomial of degree d,

$$\mathrm{Hilb}_R(t) = \frac{1}{1-t^d} = \frac{1}{d(1-t)} + \frac{d-1}{2d} + \frac{d^2-1}{12d}(1-t) + \frac{d^2-1}{24d}(1-t)^2 + \cdots .$$

- $\dim(R) = 1$,
- $\gamma_0 = \frac{1}{d}$,
- $\gamma_1 = \frac{d-1}{2d}$,
- $\gamma_2 = \frac{d^2-1}{12d}$,
- $\gamma_3 = \frac{d^2-1}{24d}$, etc.

The Laurent Expansion of the Hilbert Series, Examples

For $R = \mathbb{C}[x_1 x_2, x_1^2, x_2^2] \subset \mathbb{C}[x_1, x_2]$,

$$\mathrm{Hilb}_R(t) = \frac{1+t^2}{(1-t^2)^2} = \frac{1}{2(1-t)^2} + \frac{1}{8} + \frac{1}{8}(1-t) + \frac{3}{32}(1-t)^2 + \cdots .$$

- $\dim(R) = 2$,
- $\gamma_0 = \frac{1}{2}$,
- $\gamma_1 = 0$,
- $\gamma_2 = \gamma_3 = \frac{1}{8}$,
- $\gamma_4 = \frac{3}{32}$, etc.

Molien's Formula for Finite Groups

Molien's Formula for Finite Groups

Theorem (Molien)

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{K})$ of order $|G|$. Then

$$\mathrm{Hilb}_{\mathbb{K}[x_1,\dots,x_n]^G}(t) = \frac{1}{|G|}\sum_{A\in G}\frac{1}{\det(\mathrm{Id} - At)}.$$

An elementary proof is given in Sturmfels [5, Section 2.2].
The idea is that the averaging operation

$$\frac{1}{|G|}\sum_{A\in G} A$$

is a projection onto the subspace of vectors fixed by G, so its rank=trace is the dimension of this subspace.
This is applied to each degree, i.e. on $\mathbb{K}[x_1,\dots,x_n]_j$.

Molien's Formula for Finite Groups, Example

Let $G = \{1, -1\} \subset \mathsf{GL}_1(\mathbb{R})$.
We compute

$$\begin{aligned}\mathrm{Hilb}_{\mathbb{R}[x]^G}(t) &= \frac{1}{|G|}\sum_{A\in G}\frac{1}{\det(\mathrm{Id}-At)}\\ &= \frac{1}{2}\left(\frac{1}{\det(1-t)}+\frac{1}{\det(1+t)}\right)\\ &= \frac{1}{1-t^2}.\end{aligned}$$

This is consistent with our earlier observation that

$$\mathbb{R}[x]^G = \mathbb{R}[x^2].$$

Molien's Formula for Finite Groups, Example

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and let $G = \{\mathrm{Id}, A\} \subset \mathrm{GL}_2(\mathbb{C})$.
We compute

$$\begin{aligned}
\mathrm{Hilb}_{\mathbb{C}[x]^G}(t) &= \frac{1}{2}\left(\frac{1}{\det(\mathrm{Id} - \mathrm{Id}\, t)} + \frac{1}{\det(\mathrm{Id} - At)}\right) \\
&= \frac{1}{2}\left(\frac{1}{(1-t)^2} + \frac{1}{(1-t^2)}\right) \\
&= \frac{1}{(1-t)(1-t^2)}.
\end{aligned}$$

This is consistent with our earlier observation that

$$\mathbb{C}[x]^G = \mathbb{C}[x+y, xy],$$

and that these generators are algebraically independent.

Molien's Formula for Finite Groups, Example

Let $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and let $G = \{\mathrm{Id}, B\} \subset \mathrm{GL}_2(\mathbb{R})$.
We compute

$$\begin{aligned} \mathrm{Hilb}_{\mathbb{C}[x]^G}(t) &= \frac{1}{2}\left(\frac{1}{\det(\mathrm{Id} - \mathrm{Id}\, t)} + \frac{1}{\det(\mathrm{Id} - Bt)}\right) \\ &= \frac{1}{2}\left(\frac{1}{(1-t)^2} + \frac{1}{(1-t^2)}\right) \\ &= \frac{1+t^2}{(1-t^2)^2}. \end{aligned}$$

Recall that

$$\mathbb{C}[x]^G = \mathbb{C}[x^2, y^2, xy]$$

and this Hilbert basis is not algebraically independent.

Laurent Coefficients for Finite Group Invariants

The First Laurent Coefficient for Finite Groups

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ of order $|G|$.

Using Molien's formula, we can find interpretations for the Laurent coefficients γ_j of $\mathbb{C}[x_1, \ldots, x_n]$.

For each element $A \in G$, there is a basis for $\mathbb{C}^n$ with respect to which A is a diagonal matrix

$$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Then the corresponding term in Molien's formula is

$$\frac{1}{\det(\mathrm{Id} - \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)t)} = \frac{1}{\prod_{j=1}^{n}(1 - \lambda_j t)}.$$

The First Laurent Coefficient for Finite Groups

$$\frac{1}{\det(\mathrm{Id} - \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)t)} = \frac{1}{\prod_{j=1}^{n}(1 - \lambda_j t)}$$

has a pole of order n if and only if each $\lambda_j = 1$, i.e. $A = \mathrm{Id}$, so the first term of the Laurent series is

$$\frac{1}{|G|(1-t)^n}.$$

That is,

$$\gamma_0 = \frac{1}{|G|}.$$

The Second Laurent Coefficient for Finite Groups

The term

$$\frac{1}{\det(\mathrm{Id} - \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)t)} = \frac{1}{\prod_{j=1}^{n}(1 - \lambda_j t)}$$

contributes to the degree $1 - n$ term if and only if $\lambda_j = 1$ for all but one j. Such a group element is called a **pseudoreflection**.

In this case we have the term in Molien's formula

$$\frac{1}{(1-t)^{n-1}(1-\lambda_j t)}.$$

As the Taylor series of $\frac{1}{1-\lambda_j t}$ at $t = 1$ begins with $\frac{1}{1-\lambda_j}$, and $\det(\mathrm{diag}(1, \ldots, 1, \lambda_j)) = \lambda_j$, the corresponding term in the Laurent series is

$$\frac{1}{1 - \det A}(1-t)^{1-n}.$$

The Second Laurent Coefficient for Finite Groups

Let P be the set of pseudoreflections in G, and then the second coefficient of the Laurent series is

$$\frac{1}{|G|}\sum_{A\in P}\frac{1}{1-\det A}.$$

Noting $A \in P$ if and only if $A^{-1} \in P$, we have

$$\begin{aligned}\frac{2}{|G|}\sum_{A\in P}\frac{1}{1-\det A} &= \frac{1}{|G|}\sum_{A\in P}\frac{1}{1-\det A}+\frac{1}{1-(\det A)^{-1}}\\ &= \frac{1}{|G|}\sum_{A\in P}\frac{1-\det A}{1-\det A}\\ &= \frac{|P|}{|G|}.\end{aligned}$$

The Second Laurent Coefficient for Finite Groups

Therefore

$$\gamma_1 = \frac{|P|}{2|G|}.$$

The Laurent Coefficient for Finite Groups, Example

The Hilbert series of the invariants of a certain finite subgroup $G \leq \mathrm{GL}_4(\mathbb{C})$ is

$$\mathrm{Hilb}_{\mathbb{C}[x_1,x_2]^G}(t) = \frac{-t^{14} - 6t^{10} - 7t^8 + 7t^6 + 6t^4 + 1}{(1-t^2)(1-t^4)^2}.$$

The Laurent expansion begins

$$\mathrm{Hilb}_{\mathbb{C}[x_1,x_2]^G}(t) = \frac{1}{8}(1-t)^{-4} + \frac{25}{128} + \frac{25}{64}(1-t) + \cdots .$$

We can conclude that $|G| = 8$ and G contains no pseudoreflections.

The Hilbert Series of (Symplectic) Circle Quotients

The Molien-Weyl Formula for Compact Lie groups

If $G \leq \mathrm{GL}_n(\mathbb{K})$ is a compact Lie group (i.e. closed and bounded), then Molien's formula

$$\mathrm{Hilb}_{\mathbb{K}[x_1,\ldots,x_n]^G}(t) = \frac{1}{|G|}\sum_{A\in G}\frac{1}{\det(\mathrm{Id}-At)}$$

generalizes to the **Molien-Weyl formula**:

$$\mathrm{Hilb}_{\mathbb{K}[x_1,\ldots,x_n]^G}(t) = \int_G \frac{d\mu}{\det(\mathrm{Id}-At)}.$$

Here, $d\mu$ indicates that we integrate over the group with a normalized **Haar measure**, i.e. an invariant measure such that $\int_G d\mu = 1$.

The Molien-Weyl Formula for the Circle

When $G = \mathbb{S}^1$ is the circle and $\mathbb{K} = \mathbb{C}$, this yields

$$\mathrm{Hilb}_{\mathbb{C}[x_1,\ldots,x_n]^G}(t) = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z\det(\mathrm{Id} - zt)}.$$

If the circle has weight matrix $(a_1, \ldots, a_n)$, then this formula is

$$\mathrm{Hilb}_{\mathbb{C}[x_1,\ldots,x_n]^G}(t) = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z\prod_{j=1}^{n}(1 - z^{a_j} t)}.$$

The Molien-Weyl Formula for Symplectic $\mathbb{S}^1$-Quotients

If the circle has weight matrix $(a_1, \dots, a_n)$, then the Hilbert series of the real polynomial invariants is given by

$$\mathrm{Hilb}_{\mathbb{R}[x_1,\dots,x_n,\overline{x_1},\dots,\overline{x_n}]^G}(t) = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z \prod_{j=1}^{n} (1 - z^{a_j} t)(1 - z^{-a_j} t)}.$$

Because the shell is particularly simple in this case (the moment map components are invariants), the Hilbert series of the regular functions on the symplectic quotient is given by

$$\mathrm{Hilb}_{M_0}(t) = \frac{(1-t^2)}{2\pi i} \int_{|z|=1} \frac{dz}{z \prod_{j=1}^{n} (1 - z^{a_j} t)(1 - z^{-a_j} t)}.$$

Symplectic $\mathbb{S}^1$-Quotient: Generic Case

Assuming $|t| < 1$, the poles of

$$\frac{1}{z \prod_{j=1}^{n} (1 - z^{a_j} t)(1 - z^{-a_j} t)} = \frac{z^{-1+\sum_{j=1}^{n} a_j}}{\prod_{j=1}^{n} (1 - z^{a_j} t)(z^{a_j} - t)}$$

occur when $z^{a_j} = t$ for some j.

Hence there are $\sum_{j=1}^{n} |a_j|$ residues to consider.

If the $|a_j|$ are pairwise distinct (the **generic case**), then each pole is simple.

The sum of the residues is

$$\mathrm{Hilb}_{M_0}(t) = \sum_{j=1}^{n} \sum_{\zeta^{a_j}=1} \frac{1}{|a_j| \prod_{\substack{k=1 \\ k \neq j}}^{n} (1 - \zeta^{a_k} t^{(a_j+a_k)/a_j})(1 - \zeta^{a_k} t^{(a_j-a_k)/a_j})}.$$

Symplectic $\mathbb{S}^1$-Quotient: Degenerate Case

If the $|a_j|$ are not pairwise distinct (the **degenerate case**), then things are more complicated.

However, it can be shown that the apparent singularities in the expression for the Hilbert series are in fact removable:

$$\mathrm{Hilb}_{M_0}(t) = \lim_{(c_i)\to(a_i)} \sum_{j=1}^{n} \sum_{\zeta^{a_j}=1} \frac{1}{|c_j| \prod_{\substack{k=1\\k\neq j}}^{n} (1-\zeta^{a_k} t^{(c_j+c_k)/c_j})(1-\zeta^{a_k} t^{(c_j-c_k)/c_j})}.$$

Applications of the Formula: Laurent Coefficients

For the regular functions on the symplectic quotient, setting $\alpha_j = |a_j|$ and $g_r = \gcd\{\alpha_j : j \neq r\}$,

$$\gamma_0 = \sum_{j=1}^{n} \frac{\alpha_j^{2n-3}}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} (\alpha_j^2 - \alpha_k^2)}, \qquad \gamma_1 = 0,$$

$$\gamma_2 = \gamma_3 = \frac{-1}{12} \sum_{j=1}^{n} \frac{\alpha_j^{2n-5}}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} (\alpha_j^2 - \alpha_k^2)} \sum_{\substack{p=1\\p\neq j}}^{n} \alpha_p^2 + \frac{1}{12} \sum_{r=1}^{n} \sum_{\substack{j=1\\j\neq r}}^{n} \frac{\alpha_j^{2n-5}(g_r^2 - 1)}{\prod\limits_{\substack{k=1\\k\neq j,r}}^{n} (\alpha_j^2 - \alpha_k^2)}.$$

The apparent singularities in the degenerate case are removable.
In fact, each sum over j can be expressed as a **Schur polynomial** divided by $\prod \alpha_j + \alpha_k$.

An Algorithm for the Hilbert Series of Symplectic Circle Quotients

Motivation

$$\mathrm{Hilb}_{M_0}(t) = \sum_{j=1}^{n} \sum_{\zeta^{a_j}=1} \frac{1}{|a_j| \prod_{\substack{k=1 \\ k\neq j}}^{n} (1-\zeta^{a_k} t^{(a_j+a_k)/a_j})(1-\zeta^{a_k} t^{(a_j-a_k)/a_j})}$$

$$= \sum_{j=1}^{n} \frac{1}{|a_j|} \sum_{\zeta^{a_j}=1} \frac{1}{\prod_{\substack{k=1 \\ k\neq j}}^{n} (1-(\zeta \sqrt[a_j]{t})^{a_j+a_k})(1-(\zeta \sqrt[a_j]{t})^{a_j-a_k})}.$$

For each j, the corresponding term is given by applying the transformation

$$(U_{a_j} F)(t) = \frac{1}{a_j} \sum_{\zeta^{a_j}=1} F(\sqrt[a_j]{t}) \qquad \text{to}$$

$$F_j(t) = \frac{1}{\prod_{\substack{k=1 \\ k\neq j}}^{n} (1-t^{a_j+a_k})(1-t^{a_j-a_k})}.$$

Motivation

If $F(t) = \sum_{r=0}^{\infty} c_r t^r$, then

$$(U_a F)(t) = \sum_{r=0}^{\infty} c_{ra} t^r.$$

Example

When $a = 3$,

$$(U_3 F)(t) = c_0 + c_3 t + c_6 t^2 + c_9 t^3 + \cdots .$$

Description of the Algorithm

For each j:

- Express $F_j(t) = \frac{1}{\prod_{k \neq j}(1-t^{a_j+a_k})(1-t^{a_j-a_k})}$ as a monomial over a product of factors of the form $1 - t^m$.
- The denominator $D_j(t)$ of $(U_{a_j} F_j)(t)$ is formed by applying

$$1 - t^m \longmapsto (1 - t^{\mathrm{lcm}(a_j, m)/a_j})^{\gcd(a_j, m)}.$$

- Find the Taylor expansion of $F_j(t)$ up to degree $a_j(d_j + 2)$ where d_j is the degree of the denominator of $F_j(t)$.
- Apply U_{a_j} to this Taylor series and call the result $P_j(t)$.
- Define $N_j(t)$ to be the terms of degree at most d_j in $P_j(t)D_j(t)$.

The Hilbert series is given by

$$\sum_{j=1}^{n} \frac{N_j(t)}{D_j(t)}.$$

Example of the Algorithm: $(1, 2, 3)$

Suppose the weight matrix is $(1, 2, 3)$.

$$
\begin{aligned}
F_1(t) &= \frac{1}{(1-t^{-1})(1-t^3)(1-t^{-2})(1-t^4)} \\
&= \frac{t^3}{(1-t)(1-t^2)(1-t^3)(1-t^4)}, \\
F_2(t) &= \frac{1}{(1-t)(1-t^3)(1-t^{-1})(1-t^5)} \\
&= \frac{-t}{(1-t)^2(1-t^3)(1-t^5)}, \\
F_3(t) &= \frac{1}{(1-t)(1-t^2)(1-t^4)(1-t^5)}.
\end{aligned}
$$

Example of the Algorithm: $(1, 2, 3)$

As $a_1 = 1$, $\frac{N_j(t)}{D_j(t)} = F_1(t)$.

For $j = 2$,

$$F_2(t) = \frac{-t}{(1-t)^2(1-t^3)(1-t^5)}$$

the new denominator is

$$D_2(t) = (1-t)^2(1-t^3)(1-t^5).$$

It doesn't change, as all the powers of t are relatively prime to $a_2 = 2$.

Example of the Algorithm: $(1, 2, 3)$

We compute the Taylor series of $F_2(t)$ up to degree $a_2(d_2 + 2) = 24$:

$$- t - 2t^2 - 3t^3 - 5t^4 - 7t^5 - 10t^6 - 14t^7 - 18t^8 - 23t^9 - 29t^{10} - 36t^{11} - 44t^{12} - 53t^{13} - 63t^{14} - 74t^{15} - 87t^{16} - 101t^{17} - 116t^{18} - 133t^{19} - 151t^{20} - 171t^{21} - 193t^{22} - 216t^{23} - 241t^{24},$$

and apply U_2:

$$- 2t - 5t^2 - 10t^3 - 18t^4 - 29t^5 - 44t^6 - 63t^7 - 87t^8 - 116t^9 - 151t^{10} - 193t^{11} - 241t^{12}.$$

Example of the Algorithm: $(1, 2, 3)$

We then multiply by $D_2(t)$:

$$-2t - t^2 - 2t^3 - t^4 - 2t^5 + 297t^{13} - 233t^{14} + 8t^{15} - 287t^{16} + 242t^{17}$$
$$-295t^{18} + 234t^{19} - 6t^{20} + 289t^{21} - 241t^{22}$$

and select only the terms of degree at most $d_2 = 10$:

$$N_2(t) = -2t - t^2 - 2t^3 - t^4 - 2t^5.$$

Then

$$\frac{N_2(t)}{D_2(t)} = \frac{-2t - t^2 - 2t^3 - t^4 - 2t^5}{(1-t)^2(1-t^3)(1-t^5)}.$$

Example of the Algorithm: $(1, 2, 3)$

Applying the same process for $F_3(t)$ yields

$$\frac{N_3(t)}{D_3(t)} = \frac{1 + t + 4t^2 + 5t^3 + 5t^4 + 5t^5 + 4t^6 + t^7 + t^8}{(1-t)(1-t^2)(1-t^4)(1-t^5)}.$$

The Hilbert series is

$$\frac{N_1(t)}{D_1(t)} + \frac{N_2(t)}{D_2(t)} + \frac{N_3(t)}{D_3(t)} = \frac{1 + t^2 + 3t^3 + 4t^4 + 4t^5 + 4t^6 + 3t^7 + t^8 + t^{10}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)}.$$

Thank you!

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