§2 SINGULAR SYMPLECTIC REDUCTION AND REGULAR FUNCTIONS ON SYMPLECTIC QUOTIENTS

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2.1. HAMILTONIAN MECHANICS

2.1.1. DEFINITION Let $(A, \cdot, 1)$ be a commutative algebra with unit 1.

A Poisson bracket is a $\mathbb{R}$-bilinear operation

$$\{,\} : A \times A \longrightarrow A$$
$$(a, b) \mapsto \{a, b\}$$

such that

(1) $\{a, b\} = -\{b, a\}$ (antisymmetry)

(2) $\{a, bc\} = \{a, b\}c$ (Leibniz rule)

(3) $\{a, \{b, c\}\} + \text{cyclic}(a, b, c) = 0$

(Jacobi-identity)

EXAMPLE

(a) $\mathbb{R}^n \oplus (\mathbb{R}^n)^* = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ cotangent bundle of $\mathbb{R}^n$

(q, p)

$q = (q^1, \dots, q^n)$, $p = (p_1, \dots, p_n)$

$A := C^\infty(\mathbb{R}^{2n})$

$$\{a, b\} = \sum_{i=1}^{n} \frac{\partial a}{\partial q^i}\frac{\partial b}{\partial p_i} - \frac{\partial b}{\partial p_i}\frac{\partial a}{\partial q^i}$$

REMARKS

(i) $\{,\}$ is uniquely defined by the brackets of the coordinate functions

$\{q^i, p_j\} = \delta_{ij}$, $\{q^i, q^j\} = 0 = \{p_i, p_j\}$

(ii) Let $\phi = (\phi_i^j) \in \mathbb{R}^{n\times n}$ be invertible.
The transformed coordinates are:

$$\tilde{q}^j = \sum_i \phi^j_i \, q^i$$

$$\tilde{p}_k = \sum_k (\phi^{-1})_k^\ell \, p_\ell$$

$$\{\tilde{q}^j, \tilde{p}_k\} = \sum_{i,k} \phi^j_i (\phi^{-1})^\ell_k \{q^i, p_\ell\}$$

$$= \sum_{i,k} \phi^j_i (\phi^{-1})^\ell_k \, \delta^i_\ell$$

$$= \sum_i \phi^j_i (\phi^{-1})^i_k = \delta^j_k$$

$\Rightarrow \{,\}$ is independent of the choic of linear coordinates.

(iii) The algebra of polinomial functions $\mathbb{R}[\mathbb{R}^{2n}] = \mathbb{R}[q^1, \dots, q^n, p_1, \dots, p_n]$ is a Poisson subalgebra of $A = C^\infty(\mathbb{R}^{2n})$.

(b) Complex coordinates
$$z_i = q_i + \sqrt{-1}\, p^i$$
$$\bar{z}_i = q_i - \sqrt{-1}\, p^i$$

provide an $\mathbb{R}$-linear isomorphism
$$\mathbb{R}^{2n} \longrightarrow \mathbb{C}^n.$$

$A = C^\infty(\mathbb{C}^n)$ = smooth functions of $z_1, \dots, z_n$, $\bar{z}_1, \dots, \bar{z}_n$ that are invariant under complex conjugation

The induced Poisson bracket on $A = C^\infty(\mathbb{C}^n)$ is determined by

$$\{z_i, \bar{z}_j\} = \frac{2}{\sqrt{-1}} \delta_{ij}, \quad \{z_i, z_j\} = 0 = \{\bar{z}_j, \bar{z}_i\}.$$

(c) <u>Generalization</u>: Let $\pi = (\pi_{ij}) \in C^\infty(\mathbb{R}^n)^{n\times n}$ be antisymmetric: $\pi_{ij} = -\pi_{ji}$. Then

$$\{a, b\} := \sum_{i,j} \pi_{ij} \frac{\partial a}{\partial x^i} \frac{\partial b}{\partial x^j}$$

defines a Poisson bracket on $A = C^\infty(\mathbb{R}^n)$

$$\Longleftrightarrow \sum_j \pi_{ij} \frac{\partial \pi_{kl}}{\partial x^j} + \mathrm{cycl}(i,k,l) = 0 \quad \forall i,k,l$$

If $\pi = (\pi_{ij})$ is invertible, $\{,\}$ is called <u>symplectic</u>.

2.1.2. DEFINITION

Let $(A, \cdot, 1, \{,\})$ be a Poisson algebra and $a \in A$.

Then $\{a, \ \} =: X_a$ is called the Hamiltonian vector field associated to a.

REMARK

(i) For $A = C^\infty(\mathbb{R}^{2n})$ and $H \in A$

the vector field X_H determines a

flow $\underbrace{\dot{f} = \{H, f\} = X_H(f).}_{\text{Hamiltonian equations of motion}}$

2.2. LIE ALGEBRAS

2.2.1 DEFINITION

A Lie algebra is an $\mathbb{R}$-vector space $\mathfrak{g}$ with an $\mathbb{R}$-bilinear operation

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (\xi, \eta) \mapsto [\xi, \eta]$$

such that

(1) $[\xi, \eta] = -[\eta, \xi] \quad \forall \xi, \eta \in \mathfrak{g}$

(2) $[\xi, [\eta, \zeta]] + \text{cycl.}(\xi, \eta, \zeta) = 0 \quad \forall \xi, \eta, \zeta \in \mathfrak{g}$

EXAMPLES

(a) If $(A, \cdot, 1, \{,\})$ is a Poisson algebra then $(A, \{,\})$ is a Lie algebra.

(b) Let $G \subseteq \mathbb{R}^{s \times s}$ be a matrix Lie group. Then

$$\mathfrak{g} := \{ \dot{\gamma}(0) \in \mathbb{R}^{s \times s} \mid \exists \text{ smooth curve } \gamma(t) \subseteq G, \gamma(0) = \mathbb{1} \}$$

$$= \{ X \in \mathbb{R}^{s \times s} \mid e^{tX} \in G \}$$

is a Lie algebra with bracket

$$[X, Y] = X \circ Y - Y \circ X.$$

EXAMPLES

(i) $G = S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

$$\begin{array}{c} \| \\ e^{\sqrt{-1}\,t} \end{array}$$

$$\frac{d}{dt}\Big|_{t=0} \left(e^{2\pi \sqrt{-1}\,\alpha t} \right) = 2\pi \sqrt{-1}\,\alpha \Rightarrow \mathfrak{g} = \sqrt{-1}\,\mathbb{R} \subseteq \mathbb{C}, \quad [\,,\,] = 0$$

(ii) $G = SU_n = \{ A \in \mathbb{C}^{2 \times 2} \mid A^* A = 1 \}$
det(A) = 1

$$\det(e^{tX}) = 1$$
$$= e^{\operatorname{tr} X}$$

$$e^{tX} \in U_n$$
$$= \mathbb{1} + tX \mod (t^2)$$

$$\Rightarrow \left(e^{tX} \right)^* e^{tX} = 1 \qquad \Rightarrow X^* = -X$$
$$= \mathbb{1} + tX^* + tX \mod (t^2)$$

$\Rightarrow \mathfrak{g} = \{$ antihermitean $n \times n$ matrices with $\operatorname{tr} X = 0 \}$

(iii) $G = O_n$ (or SO_n)

$$= \{ O \in \mathbb{R}^{n\times n} \mid O^t O = 1 \}$$

$\det(O) = 1$

$$e^{tX} \in O_n \Leftrightarrow (e^{tX})^t e^{tX} = \mathbb{1}$$

$\in SO_n$

$$\Leftrightarrow \mathbb{1} + tX^t + tX = \mathbb{1} \mod (t^2)$$

$$\Leftrightarrow X = -X^t$$

$\Rightarrow \mathfrak{o}_n = \mathfrak{so}_n = \{$ antisymmetric $n\times n$-matrices with real entries $\}$

2.2.2. DEFINITION A **morphism** between Lie algebras $(\mathfrak{g}, [\,,\,])$ and $(\tilde{\mathfrak{g}}, [\,,\,]^\sim)$ is a linear map $\varphi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $[\varphi(\xi), \varphi(\eta)]^\sim = \varphi([\xi,\eta])$ $\forall \xi, \eta$

2.3 POISSON GROUP ACTIONS

A Poisson variety (manifold) is a variety (manifold) V such that such that the regular (smooth) functions $A = A(V)$ form a Poisson algebra.

An action of a compact matrix group G on V is a group homomorphism $G \to \mathrm{Aut}(V)$ $(\mathrm{Diffeo}(V))$.

$$g \mapsto \Phi_g$$

If $\forall g$ $\quad A \to A,\ a \mapsto \Phi_g^*(a)$ preserves the bracket then the action $G \to \mathrm{Aut}(V)$ is called Poisson.

The action gives rise to a morphism of Lie algebras

$$\mathfrak{g} \to \mathrm{Vect}(V) \qquad \xi \mapsto \xi_V \quad \text{such that} \quad \xi_V\{a,b\} = \{\xi_V a, b\} + \{a, \xi_V b\}.$$

2.4. MOMENT MAPS

2.4.1. DEFINITION Let V be a Poisson variety (manifold) with a Poisson action $G \longrightarrow \mathrm{Aut}(V)$.

A *moment map* is a regular (smooth) G-equivariant map

$$J : V \longrightarrow \mathfrak{g}^* \qquad (J(v))(\xi) =: J_\xi(v)$$

such that $X_{J_\xi} = \xi_V$.

A moment map can be viewed as a Lie algebra morphism

$$J : \mathfrak{g} \longrightarrow A, \qquad \xi \longmapsto J_\xi.$$

i.e. $\{J_\xi, J_\eta\} = J_{[\xi,\eta]}$.

The locus $Z := \{v \in V \mid J_\xi(v) = 0 \ \forall \xi \in \mathfrak{g}\}$ is G-stable.

EXAMPLES

cotangent lifted case

(a) $G \longrightarrow GL_n(\mathbb{R})$

$g \longmapsto (g_i{}^j)$

$e^{t\xi} =: g(t)$

$\mathfrak{g} \longmapsto \mathrm{End}(\mathbb{R}^n)$

$\xi \longmapsto \left(\frac{d}{dt}\Big|_{t=0} e^{t\xi}\right)_i{}^j = \xi_i{}^j$

$J_\xi = \sum_i p_j \, \xi_i{}^j q^i \in \mathbb{R}[\mathbb{R}^{2n}] \subseteq C^\infty(\mathbb{R}^{2n})$

quadratic

(b) Let $G \to U(\mathbb{C}^n) = U_n$ be a unitary representation of G.

This induces a morphism of Lie algebras $\mathfrak{g} \to \mathfrak{u}_n$ = antihermitean $n \times n$-matrices

$$\xi \mapsto (\xi_{ij})$$

$$J_\xi(z, \bar{z}) = \frac{\sqrt{-1}}{2} \sum_{i,j} \xi_{ij} z_i \bar{z}_j$$

EXAMPLE

(i) $G := S^1 \times \dots \times S^1 = \mathbb{T}^\ell$ ℓ-dimensional torus

$$A := (a_{ij}) := (\overset{\mathbb{Z}^\ell}{\overset{\Downarrow}{a_1}}, \dots, \overset{\mathbb{Z}^\ell}{\overset{\Downarrow}{a_n}}) \in \mathbb{Z}^{\ell \times n}$$ matrix of weights

$$t_i = e^{2\pi\sqrt{-1}\,\xi_i}$$

$$(\xi_1, \dots, \xi_\ell) \in \mathfrak{g} = \mathbb{R}^n$$

$$(\eta_1, \dots, \eta_n) := (\xi_1, \dots, \xi_\ell) \cdot A \in \mathbb{R}^n$$

$$(t_1, \dots, t_\ell) \cdot (z_1, \dots, z_n) = (e^{2\pi\sqrt{-1}\,\eta_1} z_1, \dots, e^{2\pi\sqrt{-1}\,\eta_n} z_n)$$

$$-\|- \qquad (\bar{z}_1, \dots, \bar{z}_n) = (e^{-2\pi\sqrt{-1}} \bar{z}_1, \dots, e^{-2\pi\sqrt{-1}} \bar{z}_n)$$

$$J_i(z, \bar{z}) := J_{\xi_i}(z, \bar{z}) = \frac{1}{2} \sum_j a_{ij} z_j \bar{z}_j$$

2.5. ORBIT SPACES

Let G be a compact Lie group and $V = \mathbb{R}^n$

$G \to Gl_n(\mathbb{R}) = Gl(V)$ be a representation.

We can assume $G \to O_n(\mathbb{R})$ by averaging an arbitrary inner product $(,)$ over the Haar measure.

There exists a Hilbert basis $\varphi_1, \dots, \varphi_k \in \mathbb{R}[V]^G$, i.e. $\forall f \in \mathbb{R}[V]^G \ \exists g \in \mathbb{R}[x_1, \dots, x_k]$ such that $f = g(\varphi_1, \dots, \varphi_k)$.

We can assume that $\varphi_1, \dots, \varphi_k$ are homogeneous.

The orbit space V/G is defined to be V modulo the equivalence relation

$$v \sim v' :\Longleftrightarrow G.v = G.v'.$$

It can be shown that V/G is a Hausdorff space and that the Hilbert map

$$\varphi : V \longrightarrow \mathbb{R}^k, \; v \mapsto (\varphi_1, \dots, \varphi_k) \text{ is}$$

proper and separates orbits.

$$\begin{tikzcd}
V \arrow[rr, "\varphi = (\varphi_1, \dots, \varphi_k)"] \arrow[dr] & & X = \varphi(V) \subseteq \mathbb{R}^k \\
& V/G \arrow[ur, "\bar{\varphi}"'] &
\end{tikzcd}$$

$\bar{\varphi}$ Hilbert embedding

2.5.1 STRATIFICATION BY ORBIT TYPE

Let H be a closed subgroup of G.

Then $V_{(H)} := \{v \in V \mid \exists g \in G : G_v = gHg^{-1}\}$

where $G_v := \{g \in G \mid G.v = v\}$ is the isotropy

subgroup of V.

$V_{(H)}$ are G-stable submanifolds of V and

$(V/G)_{(H)} := V_{(H)}/G$ is a manifold.

We decompose $V/G = \bigsqcup_{(H)} V_{(H)}/G$ where

(H) ranges over conjugacy classes of closed subgroups.

2.5.2 INEQUALITIES DEFINING ORBIT SPACES

$\varphi_1, \ldots, \varphi_k$ are possibly algebraically dependent, i.e there exist possibly polinomials

"off-shell relations"

$$g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_k]$$

such that $g_j(\varphi_1, \ldots, \varphi_r) = 0 \quad \forall j = 1, \ldots, r.$

Assigning degrees to the variables

$$\deg(x_i) := \deg(\varphi_i)$$

We can assume $g_1, \ldots, g_r$ to be homogeneous.

The ring $\mathbb{R}[x_1, \ldots, x_r]/\langle g_1, \ldots, g_r \rangle$ corresponds to the Zarisky closure $\overline{X}^{\,\mathrm{Zar}}$ of $X = \varphi(V)$.

$X \subseteq \overline{X}^{zar}$ is described by polinomial inequalities. They can be constructed as follows

$$(d\varphi_i, d\varphi_j) =: a_{ij} = a_{ji} \in \mathbb{R}[V]^K.$$

Hence $a_{ij} = b_{ij} \circ \varphi$. We require $B = (b_{ij})$ with $b_{ij} \in \mathbb{R}[x_1, \dots, x_k]$ to be positive semidefinite.

EXAMPLE

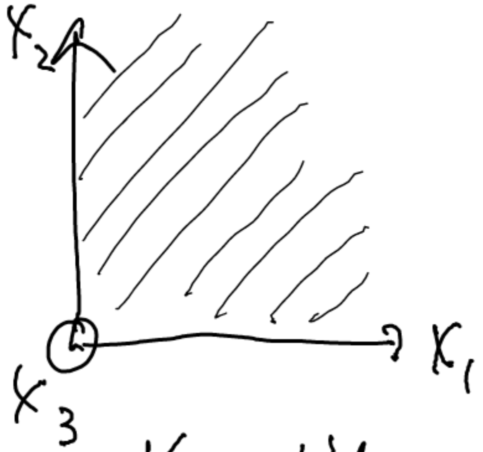
(d) $V = \mathbb{R}^{2n}$, $G = O_n$ $\boxed{n \geq 2}$

Hilbert basis: $\varphi_1 = (q, q)$, $\varphi_2 = (p, p)$

$\varphi_3 = (q, p)$

are algebraically independent.

Inequalities $x_1 \geq 0$, $x_2 \geq 0$, $x_3^2 \leq x_1 x_2$.



$X =$ solid semicone

$$(V/G)_{(\{1\})} \xrightarrow{\bar{\varphi}} \mathrm{int}(X) \subseteq \mathbb{R}^3$$

$$(V/G)_{(S^1)} \xrightarrow{\bar{\varphi}} \partial X \setminus \{0\}$$

$$(V/G)_{(O_n)} \xrightarrow{\bar{\varphi}} \{0\}$$

GENERAL FACT [Bierstone]

$\bar{\varphi}$ maps diffeomorphically orbit type strata to minimal semialgebraic strata of $X = \varphi(V)$.

2.5.3 DIFFERENTIABLE INVARIANTS [G.W. Schwarz, J. Mather]

$$C^\infty(X) = \{ f \in C(X) \mid \exists F \in C^\infty(\mathbb{R}^k) : F_{|X} = f \}$$

$$= C^\infty(\mathbb{R}^k) / I_X$$

pullback: $f \in C^\infty(X) \xmapsto[\varphi^*]{} f \circ \varphi \in C^\infty(V)^G$

TEOREM φ^* is split surjective.

2.5.4. DIFFERENTIAL SPACE

A *differential space* is a pair $(X, C^\infty(X))$ where X is a space and $C^\infty(X)$ is an algebra of continuous functions on X such that:

1º The topology of X is generated by $C^\infty(X)$.

2º $F \in C^\infty(\mathbb{R}^k)$ and $f_1, \dots, f_k \in C^\infty(X)$

$\Rightarrow F(f_1, \dots, f_k) \in C^\infty(X)$.

3º If $f : X \to \mathbb{R}$ has the property that $\forall x \in X$ exists neighborhood U of x and $f_U \in C^\infty(X)$ such that $f|_U = f_U$, then $f \in C^\infty(X)$.

If $(X, C^\infty(X))$ and $(Y, C^\infty(Y))$ are differential spaces then a smooth map between $(X, C^\infty(X))$ and $(Y, C^\infty(Y))$ is a continuous map $\varphi : X \to Y$ such that $\forall f \in C^\infty(Y)$ $\varphi^*(f) = f \circ \varphi \in C^\infty(Y)$.

A *diffeomorphism* between $(X, C^\infty(X))$ and $(Y, C^\infty(Y))$ is a homeomorphism such that $\varphi^*: C^\infty(Y) \longrightarrow C^\infty(X)$ is an isomorphism of algebras.

EXAMPLE Let $G \to O(V)$ be an orthogonal representation and $\varphi_1, \dots \varphi_k \in \mathbb{R}[V]^G$ a homogeneous Hilbert basis.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr] & X := \varphi(V) \subseteq \mathbb{R}^k \\ & V/G \arrow[u, "\bar{\varphi}"'] \end{tikzcd}$$

Then $(V/G, C^\infty(V)^G)$ and $(X, C^\infty(X))$ are differential spaces and $\bar{\varphi}: V/G \longrightarrow X$ is a diffeomorphism.

OBSERVATION

$$\bar{\varphi}^*_{|\mathbb{R}[V]^G} : \mathbb{R}[V]^G \xrightarrow{\simeq} \mathbb{R}[X] = \mathbb{R}[\bar{X}^{zar}],$$

i.e. $\bar{\varphi}^*$ sends regular functions to regular functions.

Moreover $\bar{\varphi}^*$ preserves the $\mathbb{N}$-grading.

Therefore we say that $\bar{\varphi}$ is a graded regular diffeomorphism.

REMARK It is possible to reconstruct orbit type strata from $(V/G, C^\infty(V)^G)$

(minimal strata) $\quad (X, C^\infty(X))$

2.6. SYMPLECTIC QUOTIENTS

Let V be a Poisson manifold and $G \to \mathrm{Aut}(V)$ a Poisson G-action with moment map $J : V \to \mathfrak{g}^*$ then $Z := J^{-1}(0)$ is refered to as the shell.

The symplectic quotient is

$$M_0 := V/\!\!/_0 G := Z/G.$$

[We focus on the case $V = \mathbb{C}^n$ hermitean vector space with scalar product $\langle , \rangle$ and $G \to U(V)$ is a unitary representation.]

We put $I_Z := \{ f \in C^\infty(V) \mid f_{|Z} = 0 \}$.

$$C^\infty(Z) = C^\infty(V)/I_Z$$
$$\cup\!\!\!\mid$$
$$\mathbb{R}[Z] = \mathbb{R}[V]/(I_Z \cap \mathbb{R}[V])$$

OBSERVATION OF ARMS/CUSHMAN/GOTAY

$C^\infty(M_0) := C^\infty(V)^G / I_Z \cap C^\infty(V)^G$ form a Poisson algebra and can be interpreted as the algebra of smooth functions on M_0

OBSERVATION OF SJAMAAR/LERMAN

Strata $(M_0)_{(H)} := (Z \cap V_{(H)})/G$ are symplectic manifolds.

DESCRIPTION USING INVARIANTS

$G \to U(V) \subset O(V)$

$\mathrm{Re}\langle\, , \rangle =: (\, ,)$ is Euclidean inner product

$\Rightarrow$ exist Hilbert basis $\varphi_1, \dots, \varphi_k$ and

$$\begin{tikzcd} V \arrow[rr, "\varphi = (\varphi_1, \dots, \varphi_k)"] \arrow[dr] & & \varphi(V) \subseteq \mathbb{R}^k \\ & V/G \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

For describing $M_0 = Z/G \subseteq V/G$ we need to determine the on-shell relations (inequalities remain the same).

These are given by $\ker(\Phi)$ for the homomorphism of $\mathbb{R}$-algebras:

"global chart"

$$\mathbb{R}[x_1,\dots,x_k] \xrightarrow[x_i \mapsto \varphi_i|_Z]{\Phi} C^\infty(M_0)$$

REMARK

(1) In general it is unclear whether $I_Z = \langle J_\xi \mid \xi \in \mathfrak{g} \rangle$. EXAMPLE:

$$J = \frac{1}{2} z\bar{z} \Rightarrow I_Z = \langle z, \bar{z} \rangle$$

For 1-large representations this problem does <u>not</u> occur.

(2) The case when G is abelian the determination of $\ker(\Phi)$ is way easier because J_ξ are invariant.

(3) $$\mathbb{R}[M_0] := \mathbb{R}[V]^G / I_Z \cap \mathbb{R}[V]^G$$
$$\cong \mathbb{R}[X_1, \ldots, X_k] / \ker(\Phi)$$

is an $\mathbb{N}$-graded Poisson algebra.

POISSON DIFFENTIAL SPACE

We view $(M_0, C^\infty(M_0))$ as a Poisson differential space, i.e. as a differential space with a Poisson bracket $\{\,,\}: C^\infty(M_0) \times C^\infty(M_0) \to C^\infty(M_0)$.

In addition we need:

(A) Every real maximal ideal (i.e. $\mathfrak{m} \subseteq C^\infty(X_0)$ maximal ideal s. th. $C^\infty(X_0)/\mathfrak{m} \cong \mathbb{R}$) is of the form $\mathfrak{m}_\xi = \{f \in C^\infty(X_0) \mid f(\xi) = 0\}$.

Moreover $\bigcap_{\xi \in X} \mathfrak{m}_\xi = \{0\}$

(B) Hamiltonian v.f. $D = \{h, \ \}$ for $h \in C^\infty(M_0)$ fulfill the chain rule:

For each $\varphi_1, \dots, \varphi_k \in C^\infty(M_0)$ and $F \in C^\infty(\mathbb{R}^k)$: $D(F \circ \varphi) = \sum_{i=1}^{k} \frac{\partial F}{\partial x_i} \circ \varphi \cdot D(\varphi_i)$.

THM: $(M_0, C^\infty(M_0), \{\ ,\ \})$ is a Poisson differential space fulfilling (A) and (B).

2.7. LIFTING TEOREM

Let $(M_0, C^\infty(M_0), \{,\})$ and $(N_0, C^\infty(N_0), \{,\})$ two symplectic quotients with global charts $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_k] \xrightarrow[\Phi]{} C^\infty(M_0)$

and $\mathbb{R}[y] = \mathbb{R}[y_1, \dots, y_l] \xrightarrow[\psi]{} C^\infty(N_0)$.

Assume that there is a morphism of Poisson algebras $\lambda : \mathbb{R}[y]/\ker(\psi) \to \mathbb{R}[x]/\ker \Phi$

such that $\mathrm{im}(\varphi) \supseteq \mathrm{im}(\vartheta)$ where $\varphi : Y \to \mathbb{R}^l$
$\eta \mapsto (\varphi_1(\eta), \dots, \varphi_k(\eta))$

and $\vartheta : X \to \mathbb{R}^k$, $\vartheta(\xi) = (\varphi(\lambda(y_i))(\xi))$.

Then there exists a unique smooth Poisson map $\chi : X \to Y$ such that $\chi^* = \lambda$ and

$$\begin{tikzcd} C^\infty(Y) \arrow[r, "\lambda"'] & C^\infty(X) \\ \mathbb{R}[y] \arrow[u, "\psi"] \arrow[r, "\lambda"'] & C^\infty(Y) \arrow[u, "\varphi"'] \end{tikzcd}$$

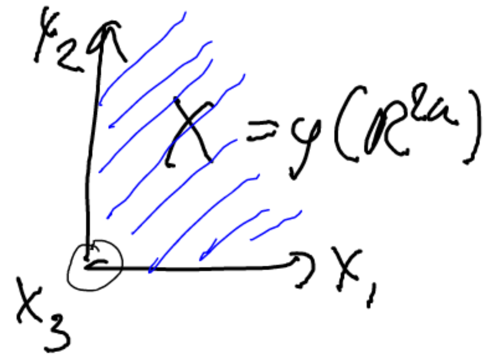
, namely

$$\chi : X \to Y, \quad \xi \mapsto \psi^{-1}(\vartheta(\xi)).$$

DEFINITION If $\lambda : \mathbb{R}[y] \to \mathbb{R}[x]$ is an isomorphism of $\mathbb{N}$-graded Poisson algebras we say $\chi : Y \to X$ is a graded regular symplectomorphism.

EXAMPLE

(d) continued $\qquad n \geq 2$



O_n acting on $\mathbb{R}^{2n}$

$O(q,p) = (Oq, Op)$

moment map $\quad J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{\binom{n}{2}} = \Lambda^2 \mathbb{R}^n$

$$(q,p) \longmapsto q \wedge p$$

$$J(q,p) = 0 \iff q \parallel p$$

$$\Rightarrow \quad Z/G \xrightarrow[\bar{\varphi}]{} \partial X$$

On shell relation:

$$\varphi_3^2 - \varphi_1 \varphi_2 = 0$$

Poisson brackets: $\varphi_1(q,p) = q\cdot q$

$\varphi_2(q,p) = p\cdot p$

$\varphi_3(q,p) = q\cdot p$

$$\{\varphi_1, \varphi_2\} = \sum_{i,j} \{q_i q_i, p_j p_j\} = 4 \sum_{i,j} q_i p_j \delta_{ij}$$

$$= 4\varphi_3$$

$$\{\varphi_1, \varphi_3\} = \sum_{i,j} \{q_i q_i, q_j p_j\}$$

$$= 2 \sum_{i,j} q_i q_j \delta_{ij} = 2\varphi_1$$

$$\{\varphi_2, \varphi_3\} = \sum_{i,j} \{p_i p_i, q_j p_j\}$$

$$= 2\sum p_i p_j (-\delta_{ij}) = -2\varphi_2$$

$\{,\}$	x_1	x_2	x_3
x_1	0	$4x_3$	$2x_1$
x_2	$\cdot$	0	$-2x_2$
x_3	$\cdot$	$\cdot$	0

$$\Rightarrow \quad \mathbb{R}[M_0] \simeq \mathbb{R}[x_1, x_2, x_3] / \langle x_3^2 - x_1 x_2 \rangle$$

$\deg x_i = 2$ Poisson ideal

is Poisson algebra of regular function.

(e) degeneration: $n=1$

$\Rightarrow V = \mathbb{R}^2, \; O_1 = \mathbb{Z}_2 \qquad (q,p) \mapsto (-q,-p)$

Hilbert basis: $\varphi_1 = q^2, \; \varphi_2 = qp, \; \varphi_3 = p^2$ P.B.

$q \parallel p$ always $\Rightarrow$ on-shell relation: the same as in (a)

$$\varphi_3 - \varphi_1 \varphi_2 = 0$$

CONCLUSION: $\mathbb{R}^{2n} /\!/_0 \, O_n$ is $\mathbb{N}$-graded symplectomorphic to $\mathbb{R}^2 / \mathbb{Z}_2$

2.8 KEMPF-NESS THEOREM

Let $G \to U(V)$ be a unitary representation. The representation extends to $G_{\mathbb{C}} \to Gl_n(\mathbb{C})$. Each closed $G_{\mathbb{C}}$-orbit meets Z in exactly one point. Moreover the corresponding map $Z \to V/\!/G_{\mathbb{C}}$ gives rise to a homeomorphism $Z/G \to V/\!/G$, the so-called Kempf-Ness homeomorphism.

EXAMPLE (d) $G_{\mathbb{C}} = O_n(\mathbb{C})$ acts on $\mathbb{R}^{2n} = \underset{z}{\mathbb{C}^n}$, $\varphi = z \cdot z$

$$V/\!/O_n(\mathbb{C}) = \mathbb{C}$$

2.9. TWO-DIMENSIONAL TORUS QUOTIENTS

PRELIMINARIES $\qquad A \in \mathbb{Z}^{\ell \times n}$ weight matrix

moment map: $J_a := \frac{1}{2} \sum_{i=1}^{n} A_{ai}\, z_i \overline{z}_i$

(1) We assume that the $\mathbb{T}^\ell$-action is faithful.

In particular $\mathrm{rk}(A) = \ell$.

(2) We assume that $0 \in \mathrm{int}(\mathrm{conv}(A))$.

$\Rightarrow \quad I_Z = \langle J_a \mid a = 1, \ldots, \ell \rangle$

(3) We can assume that $z_i \overline{z}_i =: f_i$ is part of a Hilbert basis.

$\Rightarrow \quad J_a = \frac{1}{2} \sum_{i=1}^{n} A_{ai} f_i \,.$

Assuming that $\dim_{\mathbb{R}} M_0 = 2$ one can reduce consideration to:

(4) We assume that the weight matrix is of the form

$$A = \begin{pmatrix} -a_1 & 0 & 0 & & 0 & n_1 \\ 0 & -a_2 & 0 & & 0 & n_2 \\ 0 & 0 & -a_3 & & 0 & n_3 \\ & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_e & n_e \end{pmatrix}$$

with $a_i > 0$ $\forall i = 1, \dots, n$ and $n_i \geq 0$ $\forall i = 1, \dots, e$.

Moreover, we can assume $\gcd(a_i, n_i) = 1$.

We set $\mathcal{A} := \mathrm{lcm}(a_1, \dots, a_e)$, $m_i := \dfrac{n_i \mathcal{A}}{a_i}$ and

$$\mathcal{M} := \sum_{i=1}^{e} m_i$$

The Hilbert basis can be worked out:

$$\rho_1 = \mathrm{Re}\left(z_{\ell+1}^{A} \prod_{i=1}^{\ell} z_i^{m_i} \right)$$

deg ρ_1 = deg ρ_2
$= A + M$

$$\rho_2 = \mathrm{Im}\left(z_{\ell+1}^{A} \prod_{i=1}^{\ell} z_i^{m_i} \right)$$

$$\rho_3 = z_{\ell+1} \bar{z}_{\ell+1}$$

$$\rho_4 = z_\ell \bar{z}_\ell, \ \ldots, \ \rho_{\ell+3} = z_\ell \bar{z}_\ell$$

Global chart:

$$\Phi : \mathbb{R}[y_1, \ldots, y_{\ell+3}] \longrightarrow C^\infty(M_0)$$
$$y_i \longmapsto \varphi_i := \rho_i|Z$$

$\ker(\varphi)$ is generated by:

$$y_1^2 + y_2^2 - \frac{\prod_{i=1}^{\ell} m_i^{m_i}}{A^{M}}\, y_3^{A+M}$$ "off-shell"

$$y_{3+i} - \frac{m_i}{A}\, y_3 \qquad (i = 1, \dots, \ell)$$ "on-shell"

inequalities: $y_3 \geq 0$

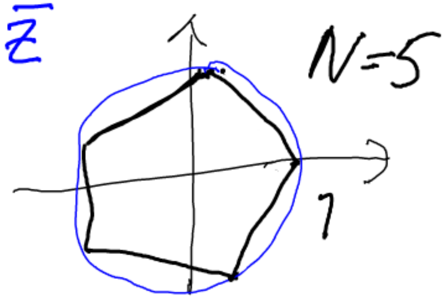
ON THE OTHER HAND: Fix $N \in \mathbb{N}$, $N \geq 2$.

Consider $\mathbb{Z}_N \subseteq S^1$.

$\mathbb{Z}_N$ is represented on $\mathbb{C}$: $g.z = gz \quad g \in \mathbb{Z}_N$

$g.\bar{z} = \bar{g}'\bar{z}$

$\boxed{g^5 = 1}$



Hilbert basis:

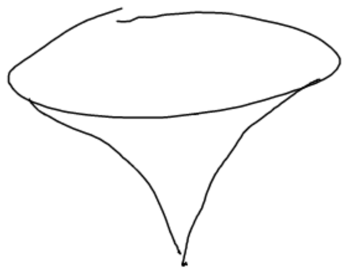
$$\varphi_1 = \mathrm{Re}(z^N), \quad \varphi_2 = \mathrm{Im}(z^N), \quad \varphi_3 = z\bar{z}$$

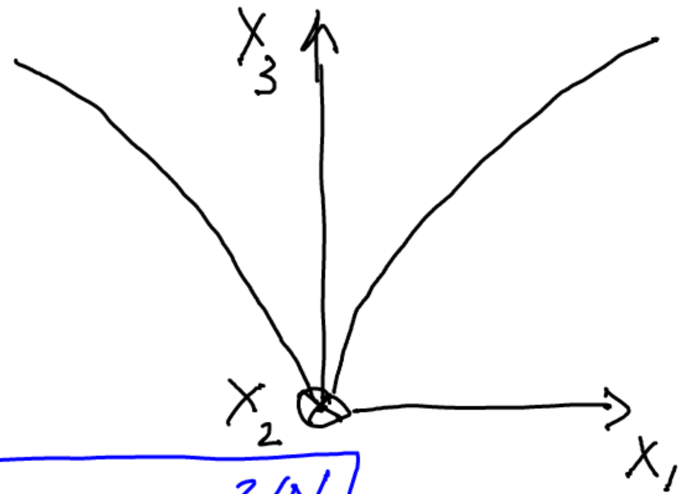
Global chart:

$$\mathbb{R}[X_1, X_2, X_3] \xrightarrow[X_i \mapsto \varphi_i]{\Phi} C^\infty(\mathbb{C})^{\mathbb{Z}_N} = C^\infty(\mathbb{C}/\mathbb{Z}_N)$$

$\mathrm{Ker}(\Phi)$ generated by: $X_1^2 + X_2^2 = X_3^N$

Inequalities: $X_3 \geq 0$





$$\boxed{X_3 = r^{2/N}}$$

2.8.1 THEOREM

With the above notation put $N := \mathcal{A} + \mathcal{M}$.

The substitution

$$y_i \longmapsto \sqrt{\frac{\mathcal{A}^{\mathcal{A}} \prod_{i=1}^{\ell} m_i^{m_i}}{N^N}}\, x_i \qquad i = 1, 2$$

$$y_3 \longmapsto \frac{\mathcal{A}}{N}\, x_3$$

$$y_{3+i} \longmapsto \frac{m_i}{N}\, x_3 \qquad i = 1, \dots, \ell$$

induces an $\mathbb{N}$-graded regular symplecto-morphism $\mathbb{C}/\mathbb{Z}_N \longrightarrow \mathbb{C}^{\ell+1} /\!\!/_0\, \mathbb{T}^{\ell}$.

PROOF $\{z_i, \bar z_j\} = \frac{2}{\sqrt{-1}} \delta_{ij}$

$$\{\rho_1, \rho_2\} = (z_{\ell+1}\bar z_{\ell+1})^{\mathcal{A}} \prod_{i=1}^{\ell} (z_i \bar z_i)^{m_i} \left(\frac{\mathcal{A}^2}{z_{\ell+1}\bar z_{\ell+1}} + \sum_{i=1}^{\ell} \frac{m_i^2}{z_i \bar z_i} \right)$$

Writing $\bar y_i$ for the class of y_i in $\mathbb{R}[y]/\ker \Phi$,

i.e. $\bar y_{i+3} = \frac{m_i}{\mathcal{A}} \bar y_3$, we gain:

$\{\bar y_1, \bar y_2\} = \mathcal{B}\, \bar y_3^{\mathcal{A}+\mathcal{M}-1}$ with $\mathcal{B} = \frac{\mathcal{A}+\mathcal{M}}{\mathcal{A}^{\mathcal{M}-1}} \prod_{i=1}^{\ell} m_i^{m_i}$.

$\{,\}$	$\bar y_1$	$\bar y_2$	$\bar y_3$
$\bar y_1$	0	$\mathcal{B}\bar y_3^{\mathcal{A}+\mathcal{M}-1}$	$2\mathcal{A}\bar y_2$
$\bar y_2$	.	0	$-2\mathcal{A}\bar y_1$
$\bar y_3$	.	.	0

omitting $\bar y_{3+i}$ $i = 1, \dots, \ell$

Similarly we write $\overline{x_i}$ for the class of x_i in $\mathbb{R}[x]/\ker(\Phi)$.

$\{,\}$	$\overline{x_1}$	$\overline{x_2}$	$\overline{x_3}$
$\overline{x_1}$	0	$N^2\overline{x_3}^{N-1}$	$2N\overline{x_2}$
$\overline{x_2}$	$\cdot$	0	$-2N\overline{x_1}$
$\overline{x_3}$	$\cdot$	$\cdot$	0

We make the Ansatz

$$y_1 \xmapsto{\lambda} \alpha x_1$$
$$y_2 \xmapsto{\lambda} \alpha x_2$$
$$y_3 \xmapsto{\lambda} \beta x_3$$

and demand $\lambda(\{y_i, y_j\}) = \{\lambda(y_i), \lambda(y_j)\}$.

It turns out that

$$2\alpha\beta N = 2A\alpha \qquad \Longrightarrow \qquad \beta = A/N$$
$$\alpha^2 N^2 = \beta^{N-1} B \qquad\qquad \alpha = \sqrt{\frac{A^A \prod_{i=1}^{\ell} m_i^{m_i}}{N^N}}$$

Due to the identity $\alpha^2 = \frac{\beta^N B}{NA}$ the generator $y_1^2 + y_2^2 - \frac{\prod_i m_i^{m_i}}{A^M} y_3^{A+M} = y_1^2 + y_2^2 - \frac{B}{NA} y_3^N$

is sent to $X_1^2 + X_2^2 - X_3^N$.

Moreover: $y_3 \geq 0 \Leftrightarrow \lambda(y_3) \geq 0$. $\square$

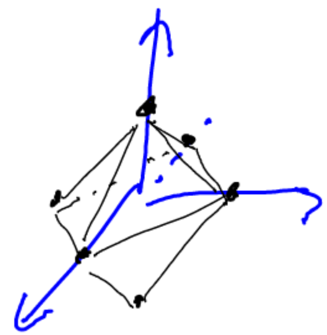
EXAMPLE

Let $A_{oct} \in \mathbb{Z}^{\ell \times 2\ell} = (e_1, -e_1, e_2, -e_2, \ldots, e_\ell, -e_\ell)$

$$= \left(\begin{array}{cc|cc|cc|c} 1 & -1 & & & & & \\ \hline & & 1 & -1 & & & \\ \hline & & & & 1 & -1 & \\ & & & & & & \ddots \end{array}\right).$$

SPECIAL CASE $\ell = 3$

$\mathrm{conv}(A) =$ octaeder



Observation: if $A = \left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right) \Rightarrow M_0^A = M_0^{A_1} \times M_0^{A_2}$.

$(-1, 1) \Rightarrow M_0^{(-1,1)} = \mathbb{Z}_2$. Hence $M_0^{A_{oct}} = (\mathbb{Z}_2)^\ell$.

2.10 SYMPLECTIC REDUCTION AT ZERO ANGULAR MOMENTUM

$k = \#$ particles

O_n-action on $\underbrace{\mathbb{R}^{2n} \oplus \cdots \oplus \mathbb{R}^{2n}}_{k\text{-times}} = V_{k,n}$

$\quad q_1, p_1 \quad \cdots \quad q_k, p_k$

$O_n \ni O$

$$O(q_1, p_1, q_2, p_2, \ldots, q_k, p_k)$$
$$= (Oq_1, Op_1, Oq_2, Op_2, \ldots, Oq_k, Op_k)$$

Alternative notation: $y_{2\ell-1} = q_\ell, \; y_{2\ell} = p_\ell$

$\ell = 1, \ldots, k$

Poisson brackets: $\mathbf{q}_\ell = (q_{\ell 1}, q_{\ell 2}, \dots, q_{\ell n})$

$\mathbf{p}_\ell = (p_{\ell 1}, p_{\ell 2}, \dots, p_{\ell n})$

$$\{q_{\ell_1,\alpha}, p_{\ell_2,\beta}\} = \delta_{\ell_1 \ell_2}\, \delta_{\alpha\beta}$$

moment map: $J : V_{k,n} \longrightarrow \sigma_n = \Lambda^2 \mathbb{R}^n$

$$J(\mathbf{q},\mathbf{p}) = \sum_{\ell=1}^{n} \mathbf{q}_\ell \wedge \mathbf{p}_\ell$$

$1 \le \alpha, \beta \le n$: $J_{\alpha,\beta}(\mathbf{q},\mathbf{p}) = J_{k,n,\alpha,\beta}(\mathbf{q},\mathbf{p}) = \sum_{\ell=1}^{n} q_{\ell\alpha} p_{\ell\beta} - q_{\ell\beta} p_{\ell\alpha}$

O_n-invariants: $\varphi_{i,j} = \langle y_i, y_j \rangle \qquad 1 \le i \le j \le 2k$

SO_n-invariants: in addition for $1 \le i_1 < i_2 < \dots < i_n \le 2k$

$$\psi_{i_1,\dots,i_n} = \det(y_{i_1}, y_{i_2}, \dots, y_{i_n})$$

$$\mathbb{R}[x_{ij} \mid 1 \le i \le j \le 2k] \xrightarrow[x_{ij} \mapsto \varphi_{ij}]{\Phi} \mathbb{R}[V_{k,n}]^{O_n}$$

$B := (x_{ij})$

$\ker \Phi =$ on-shell relations

$= \mathrm{rk}(B) \le n$

$= \langle (n+1)\times(n+1)\text{-minors of } B \rangle$

Inequalities:

$B \succeq 0$

Commutation relations: $\ell_1, \ell_2, \ell_3, \ell_4 = 1, 2, \dots, k$

$$\boxed{\begin{array}{l} i \ge j \\ x_{ij} := x_{ji} \end{array}}$$

$$\{x_{2\ell_1-1,2\ell_2-1}, x_{2\ell_3,2\ell_4}\}$$

$$= \delta_{\ell_1\ell_2} x_{2\ell_2-1,2\ell_4} + \delta_{\ell_1\ell_4} x_{2\ell_2-1,2\ell_3}$$

$$+ \delta_{\ell_2\ell_3} x_{2\ell_1-1,2\ell_4} + \delta_{\ell_2\ell_4} x_{2\ell_1-1,2\ell_3}$$

$$\{x_{2\ell_1-1,2\ell_2}, x_{2\ell_3-1,2\ell_4}\}$$
$$= \delta_{\ell_1\ell_4}\, x_{2\ell_3-1,2\ell_2} - \delta_{\ell_2\ell_3}\, x_{2\ell_1-1,2\ell_4}$$

$$\{x_{2\ell_1-1,2\ell_2}, x_{2\ell_3-1,2\ell_4-1}\}$$
$$= -\delta_{\ell_2\ell_4}\, x_{2\ell_1-1,2\ell_3-1} - \delta_{\ell_2\ell_3}\, x_{2\ell_1-1,2\ell_4-1}$$

$$\{x_{2\ell_1-1,2\ell_2}, x_{2\ell_3,2\ell_4}\}$$
$$= \delta_{\ell_1\ell_3}\, x_{2\ell_2,2\ell_4} + \delta_{\ell_1\ell_4}\, x_{2\ell_2,2\ell_3}$$

$$\{J_{\alpha\beta}, J_{\gamma\varepsilon}\} = \delta_{\alpha,\varepsilon} J_{\beta,\gamma} + \delta_{\beta,\gamma} J_{\varepsilon\alpha} + \delta_{\alpha\gamma} J_{\beta,\varepsilon} + \delta_{\beta,\varepsilon} J_{\alpha,\gamma}$$

$\Rightarrow$ für $I_J := \langle J_{\alpha\beta} \rangle \qquad \{I_J, I_J\} \subseteq I_J$

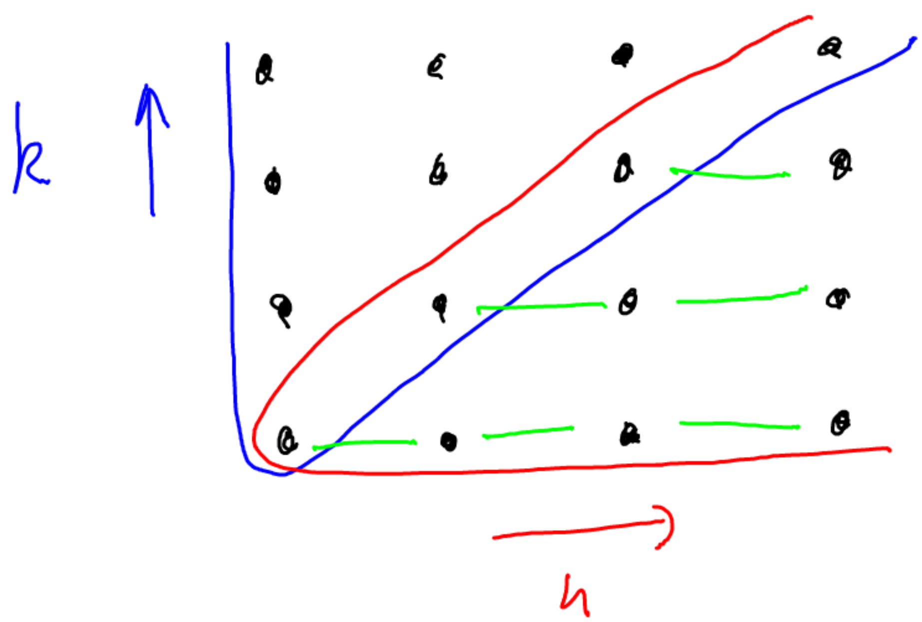
$$Q_{k,n,i,j} := Q_{i,j} := \sum_{\ell=1}^{n} \det \begin{pmatrix} x_{i,2\ell-1} & x_{i,2\ell} \\ x_{j,2\ell-1} & x_{j,2\ell} \end{pmatrix}$$

Cape

2.10.1 THEOREM

(1) $J(y) = 0 \iff Q_{i,j}(x) = 0$

(2) $(k+1)\times(k+1)$-minors of $B \subseteq \langle Q_{ij} \rangle$



$k \geq n$ "large"

$n \geq k$ "small"

$$\mathbb{R}[x_{ij}] \xrightarrow{\Phi} C^\infty(V/\!/_0 G)$$

$$\ker \Phi = \begin{cases} \sqrt[\mathbb{R}]{I_Q + I_{n+1}} & k > n \\ \sqrt[\mathbb{R}]{I_Q} & k \leq n \end{cases}$$

CONSEQUENCES OF THE THEOREM

(1) $m, n \geq k$: $V_{k,n} /\!\!/_0 O_n \simeq V_{k,m} /\!\!/ O_m$

(2) $n \geq k+1$: $V_{k,n} /\!\!/_0 O_n \simeq V_{k,n} /\!\!/_0 SO_n$

(3) $G := O_n$ or SO_n.

$V_{k,n} /\!\!/ G \simeq \mathbb{C}^r / \Gamma \iff k$ or $n = 1$

affine space mod finite group

(4) $k \geq n$: $Z_{k,n}$ has rational singularities

X has rat. sing. $\iff \exists\, Y \xrightarrow{f} X$, Y reg, proper birational such that $R^i f_* = 0 \;\; \forall i \geq 1$

TOOL: $n \le k+1$: $G = O_n$ or SO_n

$G_{\mathbb{C}} \circlearrowright V$ is 1-large

($\Rightarrow$ shell $Z_{k,n}$ irreducible, reduced, complete intersection, coherent)

$n \le k$: $SO_n(\mathbb{C}) \circlearrowright V$ is 2-large

($\Rightarrow$ shell $Z_{k,n}$ is also normal)

⚠ $O_n(\mathbb{C}) \circlearrowright V$ is not 2-large

$\boxed{\text{Macaulay2 terminal:}}$

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2.10.2. ON-SHELL RELATIONS USING ELIMINATION

Elimination: $q_j, p_j > x_i$

Let us calculate the on shell relations for $k=1$, $n=3$ using MACAULAY2:

```
i1: S = QQ[q1,q2,q3,p1,p2,p3,x1,x2,x3,
          Degrees => {1,1,1,1,1,1,2,2,2},
          MonomialOrder => Eliminate 6];

i2: I = ideal(x1-(q1*q1+q2*q2+q3*q3),
              x2-(p1*p1+p2*p2+p3*p3),
              x3-(q1*p1+q2*p2+q3*p3));
```

i3: J = I + ideal(q1*p2 - q2*p1,
q2*p3 - q3*p2,
q1*p3 - q3*p1);

i4: selectInSubring(1, gens gb J)

o4: | x1x2 - x3^2 |

i5: K = ideal(o4)

o5: $$\frac{1-T^{4}}{(1-T^2)^3\cancel{(1-T)^6}}$$

Interpretation: We cut out a degree 4 hypersurface from $\mathbb{R}[x_1, x_2, x_3]$

RESULT: $\ker(\psi) = \langle x_1 x_2 - x_3^2 \rangle$

$$\mathrm{Hilb}_{V_{1,3} /\!/_0 O_n}(t) = \frac{1-t^4}{(1-t^2)^3} = \frac{1-t^2}{(1-t^2)^2}$$

$$\ker(\psi) = \langle I_{Q_{1,1}} \rangle$$

Other cases amenable to computation:

$$\mathrm{Hilb}_{V_{2,1} /\!/_0 O_n}(t) = \frac{1+6t^2+t^4}{(1-t^2)^4}$$

$$\mathrm{Hilb}_{V_{2,2} /\!/_0 O_n}(t) = \frac{1+4t^2+4t^4+t^6}{(1-t^2)^6}$$

$$\mathrm{Hilb}_{V_{2,2} /\!/_0 SO_n}(t) = \frac{1+9t^2+9t^4+t^6}{(1-t^2)^6}$$

$$\mathrm{Hilb}_{V_{0,3,1}/\!/_0 O_1}(t) = \frac{1+15t^2+15t^4+t^6}{(1-t^2)^6}$$

In all these cases: $\ker(\Psi) = \langle I_Q \rangle$

$$\mathrm{Hilb}_{V_{0,3,2}/\!/_0 O_2}(t) = \frac{1+11t^2+51t^4+51t^6+11t^8+t^{10}}{(1-t^2)^{10}}$$

$$\mathrm{Hilb}_{V_{0,3,3}/\!/_0 O_3}(t) = \frac{1+9t^2+30t^4+44t^6+30t^8+9t^{10}+t^{12}}{(1-t^2)^{12}}$$

$$\mathrm{Hilb}_{V_{3,2}/\!/_0 SO_3}(t) = \frac{1+25t^2+100t^4+100t^6+25t^8+t^{10}}{(1-t^2)^{10}}$$

$$\mathrm{Hilb}_{V_{4,1}/\!/_0 O_1}(t) = \frac{1+28t^2+70t^4+28t^6+t^8}{(1-t^2)^8}$$

$$\mathrm{Hilb}_{V_{4,2}/\!/_0 O_2}(t) = \frac{1+22t^2+225t^4+610t^6+610t^8+225t^{10}+22t^{12}+t^{14}}{(1-t^2)^{14}}$$

In addition:

$$\mathrm{Hilb}_{Q_{4,4}}(t) = \frac{1+16t^2+108t^4+395t^6+842t^8+1080t^{10}\cdots+16t^{18}+t^{20}}{(1-t^2)^{20}}$$

$$= \mathrm{Hilb}_{V_{4,4}/\!/_0 O_4}(t)\ ?$$

Empirical observation:

$q_{1,1} \succ p_{1,1} \succ q_{1,2} \succ p_{1,2} \succ \cdots \succ q_{1,n} \succ p_{1,n} \succ q_{2,1} \succ p_{2,1} \succ \cdots \succ q_{k,n} \succ p_{k,n}$

is way better than lexicographic ordering.

e.g. $k=2, n=4$: above term ordering 1 sec

lexicographic orderin 403 sec

$D :=$ Krull dimension $\quad a := \deg(\mathrm{Hilb}(t)) =$ a-invariant

CONJECTURE:

$\forall k,n: \quad \overline{V_{k,n} /\!/_0 \, O_n}^{\,\mathrm{Zar}}$ is Gorenstein

such that $D + a = 0$.

Gorenstein means: Cohen-Macaulay

such that $\mathrm{Hilb}(t^{-1}) = (-1)^D t^{-a}\, \mathrm{Hilb}(t)$.