# Computations from the paper: *The spectra of digraphs with Morita equivalent C*-algebras* arXiv:2010.10769 [math.CO] v1

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This notebook contains the following; all Section references are to the paper in the title:

Functions that implement the graph moves described in Section 4.

Functions that compute the matrices and spectra in Section 2.3.

For the spectra and some related functions, there are two versions, one that gives the spectrum only, and one that shows the steps for checking the computations.

Many of the graphs here are based on examples in:

Søren Eilers, Gunnar Restorff, Efren Ruiz, and Adam P. W. Sørensen, *The complete classification of unital graph C*-algebras: Geometric and strong*, arXiv:1611.07120v2 [math.OA], 2019.

Adam P. W. Sørensen, *Geometric classification of simple graph algebras*, Ergodic Theory Dynam. Systems 33 (2013), no. 4, 1199--1220. MR 3082546

See the above paper for specific citations.

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## List of Functions

**Functions defining the moves:**

isSink[graph, vertex]

checks if a vertex is a sink

isSource[graph, vertex]

checks if a vertex is a source

sInv[graph, vertex]

computes $s^{-1}$(vertex)

rInv[graph, vertex]

computes $r^{-1}$(vertex)

moveS[graph]

applies move S to graph. prompts you for the vertex to remove.

moveSnoInput[graph]

applies move S to graph, but does not prompt; always removes vertex 1 or outputs that it

isn’t a source (for automation)
moveSinv[graph]
applies the inverse of move S to graph. prompts you for the name of the new vertex and which edges to have.
moveR[graph]
applies move R to graph. prompts you for the vertex at which to reduce.
moveRnoInput[graph]
applies move R to graph, but does not prompt; always reduces at vertex 1 or outputs that it can’t be done
moveO[graph, newV]
applies move O to graph. prompts you for the vertex at which to out-split and the partition.
newV is the name for the new vertices (which are indexed by subscripts).
moveI[graph, newV]
applies move I to graph. prompts you for the vertex at which to in-split and the partition.
newV is the name for the new vertices (which are indexed by subscripts).
moveC[graph, newV]
applies move C to graph. prompts you for the vertex at which to splice.
does not test that it supports two simple loops (but reminds you to check).
newV is the name for the two new vertices (which are indexed by subscripts).

**Functions defining the matrices and spectra:**
incidenceMatrix[graph]
computes the incidence matrix of graph
laplaceSpec[graph]
computes the Laplace spectrum of graph (laplaceSpecS is the same but outputs the steps of the computation)

adjacencyMatrixBinary[graph]
computes the binary adjacency matrix of graph
adjacencySpecBinary[graph]
computes the binary adjacency spectrum of graph (__S is the same but outputs the steps of the computation)
adjacencySpecSymmetricBinary[graph]
computes the symmetric binary adjacency matrix of graph (__S is the same but outputs the steps of the computation)
adjacencyMatrix[graph]
computes the adjacency matrix of graph
adjacencySpec[graph]
computes the adjacency spectrum of graph (__S is the same but outputs the steps of the computation)
adjacencySpecSymmetric[graph]
computes the symmetric adjacency spectrum of graph (__S is the same but outputs the steps of the computation)

lineAdjacencyMatrix[graph]

computes the line adjacency matrix of graph
lineGraph[graph]
computes the line graph of graph
lineAdjacencySpec[graph]
computes the line adjacency spectrum of graph (__S is the same but outputs the steps of the computation)

hermitianAdjacencyMatrix[graph]
computes the Hermitian adjacency matrix of graph
hermitianAdjacencySpectrum[graph]
computes the Hermitian spectrum of graph (__S is the same but outputs the steps of the computation)

skewAdjacencyMatrixBinary[graph]
computes the binary skew adjacency matrix of graph
skewAdjacencySpecBinary[graph]
computes the binary skew adjacency spectrum of graph (__S is the same but outputs the steps of the computation)
skewAdjacencyMatrix[graph]
computes the skew adjacency matrix of graph
skewAdjacencySpec[graph]
computes the skew adjacency spectrum of graph (__S is the same but outputs the steps of the computation)

skewLaplacianMatrix[graph]
computes the skew Laplacian matrix of graph
skewLaplaceSpec[graph]
computes the skew Laplacian spectrum of graph (__S is the same but outputs the steps of the computation)
skewLaplacianMatrixBinary[graph]
computes the binary skew Laplacian matrix of graph
skewLaplaceSpecBinary[graph]
computes the binary skew Laplacian spectrum of graph (__S is the same but outputs the steps of the computation)

transitionProbabilityMatrixBinary[graph]
computes the binary transition probability matrix of graph
normalizedLaplacianBinary[graph]
computes the binary normalized Laplacian of graph (__S is the same but outputs the steps of the computation)
combinatorialLaplacianBinary[graph]
computes the binary combinatorial Laplacian of graph (__S is the same but outputs the steps of the computation)
normalizedLaplaceSpecBinary[graph]
computes the binary normalized Laplacian spectrum of graph (__S is the same but outputs the steps of the computation)

combinatorialLaplaceSpecBinary[graph]
computes the binary combinatorial Laplacian spectrum of graph (__S is the same but outputs the steps of the computation)

transitionProbabilityMatrix[graph]
computes the transition probability matrix of graph
normalizedLaplacian[graph]
computes the normalized Laplacian of graph (__S is the same but outputs the steps of the computation)
combinatorialLaplacian[graph]
computes the combinatorial Laplacian of graph (__S is the same but outputs the steps of the computation)
normalizedLaplaceSpec[graph]
computes the normalized Laplacian spectrum of graph (__S is the same but outputs the steps of the computation)
combinatorialLaplaceSpec[graph]
computes the combinatorial Laplacian spectrum of graph (__S is the same but outputs the steps of the computation)

# Function Definitions

## Functions defining the moves (definitions are in Section 4 of arXiv:2010.10769 [math.CO] v1)

```
(* tests whether a vertex is a sink in the given graph; outputs True or False *)
isSink[g0_, v0_] := Module[{graph = g0, v = v0},
   If[MemberQ[VertexList[graph], v] == False,
    Print["The given vertex is not a vertex in the graph."],
    If[EdgeList[graph, v ↔ _] == {}, True, False]]];

(* tests whether a vertex is a source; outputs True or False *)
isSource[g0_, v0_] := Module[{graph = g0, v = v0},
   If[MemberQ[VertexList[graph], v] == False,
    Print["The given vertex is not a vertex in the graph."],
    If[EdgeList[graph, _ ↔ v] == {}, True, False]]];

(* computes s^(-1) of the given vertex. *)
sInv[g0_, v0_] := Module[{graph = g0, v = v0},
   If[MemberQ[VertexList[graph], v] == False,
    Print["The given vertex is not a vertex in the graph."],
    EdgeList[graph, v ↔ _]]];

(* computes r^(-1) of the given vertex. *)
rInv[g0_, v0_] := Module[{graph = g0, v = v0},
   If[MemberQ[VertexList[graph], v] == False,
    Print["The given vertex is not a vertex in the graph."],
    EdgeList[graph, _ ↔ v]]];
```

```
(* performs Move S (remove a source at a regular vertex).
  Takes a graph (graph).
  Prompts for the vertex to remove (vRem).
  Tests that move O can be done (the given vertex must be a regular source).
  Creates a new graph (gOut) by performing Move S.
*)
moveS[g0_] := Module[{graph = g0, vRem, gOut},
   vRem = Input[{"The vertices are ", VertexList[graph],
      ". Which vertex would you like to remove? (it must be a regular source)"}];
   If[MemberQ[VertexList[graph], vRem] == False,
    Print["The given vertex is not a vertex in the graph."];
    gOut = graph,
    Print["The vertex selected is ", vRem, "."];
    If[isSink[graph, vRem] == True,
     Print["The vertex is a sink and
        hence not regular. Move S cannot be performed here."];
     gOut = graph,
     If[isSource[graph, vRem] == False,
      Print[
       "The vertex is regular but not a source. Move S cannot be performed here."];
      gOut = graph,
      gOut = Graph[Complement[VertexList[graph], {vRem}], DeleteCases[
          EdgeList[graph], Alternatives @@ sInv[graph, vRem]], VertexLabels → "Name"];
     ]]];
   gOut];


(* same as moveS, but doesn't request input and just applies to vertex 1 *)
moveSnoInput[g0_] := Module[{graph = g0, vRem, gOut},
   vRem = 1;
   If[MemberQ[VertexList[graph], vRem] == False,
    Print["The given vertex is not a vertex in the graph."];
    gOut = graph,
    Print["The vertex selected is ", vRem, "."];
    If[isSink[graph, vRem] == True,
     Print["The vertex is a sink and
        hence not regular. Move S cannot be performed here."];
     gOut = graph,
     If[isSource[graph, vRem] == False,
      Print[
       "The vertex is regular but not a source. Move S cannot be performed here."];
      gOut = graph,
      gOut = Graph[Complement[VertexList[graph], {vRem}], DeleteCases[
          EdgeList[graph], Alternatives @@ sInv[graph, vRem]], VertexLabels → "Name"];
     ]]];
   gOut];


(* performs the inverse of Move S (adds a regular source).
  Takes a graph (graph).
  Prompts for the name of the new vertex (a source)
 and then the old vertices to which it should emit.
  Tests that the new vertex is not already a vertex
 and will be a source based on user input.
```

```
  Creates a new graph (gOut) by performing the inverse of Move S.
*)
moveSinv[g0_] := Module[{graph = g0, vAdd, edges, gOut},
   vAdd = Input[{"The vertices are ",
      VertexList[graph], ". What shall we call the new vertex?"}];
   If[MemberQ[VertexList[graph], vAdd],
    Print["The new vertex cannot already be a vertex of the graph."];
    gOut = graph,
    edges = Input[{"The new vertex ", vAdd,
       " must be a regular source. The other vertices are ", VertexList[graph],
       ". List the vertices to which ", vAdd, " will emit in brackets {}
         separated by commas, i.e. {vertex1, vertex2,...}."}];
    If[MemberQ[edges, vAdd],
     Print["The new vertex must be a source so cannot emit to itself."];
     gOut = graph,
     If[edges == {},
      Print["The new vertex must be a regular source so should emit to something.
         I'll add it, but the result won't be connected"]];
     gOut = Graph[Join[VertexList[graph], {vAdd}], Join[EdgeList[graph],
        Table[vAdd ↔ x, {x, edges}] ], VertexLabels → "Name"];
    ]];
   gOut
  ];


(* performs move R (reduction at a regular vertex).
     Takes a graph (graph).
     Prompts for the vertex to be reduced.
     Tests that it satisfies the hypotheses:
     a regular vertex u with s^-1(u) a 1-point set and s(r^-1(u)) a 1-point set).
  Creates a new graph (gOut) by performing Move R. *)
moveR[g0_] := Module[{graph = g0, vRem, gOut, sri, edges, v, k, e},
   vRem = Input[{"The vertices are ", VertexList[graph],
      ". Which vertex would you like to remove? (it must be a regular vertex
        u with s^-1(u) a 1-point set and s(r^-1(u)) a 1-point set)"}];
   If[MemberQ[VertexList[graph], vRem] == False,
    Print["The given vertex is not a vertex in the graph."];
    gOut = graph,
    If[Length[sInv[graph, vRem]] ≠ 1,
     Print["s^-1(u) is not a 1-point set"];
     gOut = graph,
     (* list all edges pointing into vRem *)
     edges = rInv[graph, vRem];
     (* find all vertices with edges that point into vRem,
     source of range^(-1) of vRem *)
     sri = Union[Table[edges[[k]][[1]], {k, 1, Length[edges]}]];
     If[Length[sri] ≠ 1,
      Print["s(r^-1(u)) is not a 1-point set"];
      gOut = graph,
      (* define v, the only vertex that emits to vRem *)
      v = sri[[1]];
      gOut = Graph[Complement[VertexList[graph], {vRem}],
        Join[ DeleteCases[EdgeList[graph],
          Alternatives @@ Join[rInv[graph, vRem], sInv[graph, vRem]] ],
```

```
          Table[v ⟷ sInv[graph, vRem][[1]][[2]], {x, rInv[graph, vRem]}]],
         VertexLabels → "Name"];
      ]]];
    gOut
   ];


(* same as moveR, but doesn't request input and just applies to vertex 1 *)
moveRnoInput[g0_] := Module[{graph = g0, vRem, gOut, sri, edges, v, k, e},
    vRem = 1;
    If[MemberQ[VertexList[graph], vRem] == False,
     Print["The given vertex is not a vertex in the graph."];
     gOut = graph,
     If[Length[sInv[graph, vRem]] ≠ 1,
      Print["s^-1(u) is not a 1-point set"];
      gOut = graph,
      (* list all edges pointing into vRem *)
      edges = rInv[graph, vRem];
      (* find all vertices with edges that point into vRem,
      source of range^(-1) of vRem *)
      sri = Union[Table[edges[[k]][[1]], {k, 1, Length[edges]}]];
      If[Length[sri] ≠ 1,
       Print["s(r^-1(u)) is not a 1-point set"];
       gOut = graph,
       (* define v, the only vertex that emits to vRem *)
       v = sri[[1]];
       gOut = Graph[Complement[VertexList[graph], {vRem}],
         Join[ DeleteCases[EdgeList[graph],
           Alternatives @@ Join[rInv[graph, vRem], sInv[graph, vRem]] ],
          Table[v ⟷ sInv[graph, vRem][[1]][[2]], {x, rInv[graph, vRem]}]],
         VertexLabels → "Name"];
      ]]];
    gOut
   ];


(* performs move O (out-split the graph).
    Takes a graph (graph) and a name for the new vertices (newV).
    Prompts for the vertex at which to out-split.
    Tests that it satisfies the hypotheses, i.e., it is not a sink.
   Prompts the user for the partition.
   Creates a new graph (gOut) by performing Move O. *)
moveO[g0_, newV_] :=
  Module[{graph = g0, vSplit, gOut, siv, n, newVerts, i, edges, j, pnum, newEdges, part},
    vSplit = Input[{"The vertices are ", VertexList[graph],
       ". Which vertex would you like to out-split? (it cannot be a sink)"}];
    If[MemberQ[VertexList[graph], vSplit] == False,
     Print["The given vertex is not a vertex in the graph."];
     gOut = graph,
     If[isSink[graph, vSplit] == True,
      Print["This vertex is a sink, and cannot be out-split."];
      gOut = graph,
      siv = sInv[graph, vSplit];
      n = Input[{"s^-1(v) is ", siv,
```

```
        "Enter n, the number of sets in the partition. The maximum value is",
        Length[siv]}];
    edges = Table[{EdgeList[graph][[i]], 0}, {i, 1, Length[EdgeList[graph]]}];
    j = 1;
    While[j ≤ Length[edges],
     If[MemberQ[siv, edges[[j, 1]] ],
      pnum = Input[{"s^-1(v) is ", siv,
         " We will now define the partition. I will display each edge individually.
           Please enter the number of the partition to which it
           belongs. Enter the partition number for the edge ",
         edges[[j, 1]], " a number from 1 to ", n}];
      If[MemberQ[Table[i, {i, 1, n}], pnum],
       edges[[j, 2]] = pnum,
       Print["The partition number is not valid so the
          output will not make sense; please start over"]]];
     j++];
    newVerts = Table[newV_i, {i, 1, n}];
    newEdges = {};
    j = 1;
    While[j ≤ Length[edges],
     part = edges[[j, 2]];
     If[edges[[j, 1]][[2]] === vSplit,
      If[edges[[j, 1]][[1]] === vSplit,
       newEdges = Join[newEdges, Table[newV_part ↔ newV_i, {i, 1, n}]],
       newEdges = Join[newEdges, Table[edges[[j, 1]][[1]] ↔ newV_i, {i, 1, n}]];
       ],
      If[edges[[j, 1]][[1]] === vSplit,
        AppendTo[newEdges, newV_part ↔ edges[[j, 1]][[2]] ],
        AppendTo[newEdges, edges[[j, 1]] ];
       ];
     ];
     j++];
    gOut = Graph[Join[Complement[VertexList[graph], {vSplit}], newVerts],
      newEdges, VertexLabels → "Name"];
   ]];
  gOut
 ];

(* performs move I (in-split the graph).
   Takes a graph (graph) and a name for the new vertices (newV).
   Prompts for the vertex at which to in-split.
   Tests that it satisfies the hypotheses, i.e., it is not a source.
  Prompts the user for the partition.
  Creates a new graph (gOut) by performing Move I. *)
moveI[g0_, newV_] :=
  Module[{graph = g0, vSplit, gOut, riv, n, newVerts, i, edges, j, pnum, newEdges, part},
   vSplit = Input[{"The vertices are ", VertexList[graph],
      ". Which vertex would you like to in-split? (it cannot be a source)"}];
   If[MemberQ[VertexList[graph], vSplit] == False,
    Print["The given vertex is not a vertex in the graph."];
    gOut = graph,
    If[isSource[graph, vSplit] == True,
     Print["This vertex is a source, and cannot be in-split."];
     gOut = graph,
```

```
      riv = rInv[graph, vSplit];
      n = Input[{"r^-1(v) is ", riv,
         "Enter n, the number of sets in the partition. The maximum value is",
         Length[riv]}];
      edges = Table[{EdgeList[graph][[i]], 0}, {i, 1, Length[EdgeList[graph]]}];
      j = 1;
      While[j ≤ Length[edges],
       If[MemberQ[riv, edges[[j, 1]] ],
        pnum = Input[{"r^-1(v) is ", riv,
           " We will now define the partition. I will display each edge individually.
             Please enter the number of the partition to which it
             belongs. Enter the partition number for the edge ",
           edges[[j, 1]], " a number from 1 to ", n}];
        If[MemberQ[Table[i, {i, 1, n}], pnum],
         edges[[j, 2]] = pnum,
         Print["The partition number is not valid so the
            output will not make sense; please start over"]]];
       j++];
      newVerts = Table[newV_i, {i, 1, n}];
      newEdges = {};
      j = 1;
      While[j ≤ Length[edges],
       part = edges[[j, 2]];
       If[edges[[j, 1]][[2]] === vSplit,
        If[edges[[j, 1]][[1]] === vSplit,
         newEdges = Join[newEdges, Table[newV_i ↔ newV_part, {i, 1, n}]],
         AppendTo[newEdges, edges[[j, 1]][[1]] ↔ newV_part ];
        ],
        If[edges[[j, 1]][[1]] === vSplit,
          newEdges = Join[newEdges, Table[newV_i ↔ edges[[j, 1]][[2]], {i, 1, n}]],
          AppendTo[newEdges, edges[[j, 1]] ];
         ];
       ];
       j++];
      gOut = Graph[Join[Complement[VertexList[graph], {vSplit}], newVerts],
        newEdges, VertexLabels → "Name"];
     ]];
    gOut
   ];

(* performs move C (Cunz split).
  Takes a graph (graph) and a name for the two new vertices (newV).
  Prompts for the vertex at which to perform the splice.
  Does not test that the vertex supports two simple loops.
  Creates a new graph (gOut) by performing Move C. *)
moveC[g0_, newV_] := Module[{graph = g0, vSplice, gOut},
   vSplice = Input[{"The vertices are ", VertexList[graph],
      ". At which vertex would you like to Cuntz splice? Note that this
        vertex must support two simple loops, and you need to check
        this yourself, as this will perform the move regardless."}];
   If[MemberQ[VertexList[graph], vSplice] == False,
    Print["The given vertex is not a vertex in the graph."];
    gOut = graph,
    gOut = Graph[ Join[Complement[VertexList[graph], {vSplice}], {newV_1, newV_2} ],
```

```
       Join[EdgeList[graph], {vSplice ↔ newV1, newV1 ↔ vSplice,
         newV1 ↔ newV1, newV1 ↔ newV2, newV2 ↔ newV1, newV2 ↔ newV2}],
       VertexLabels → "Name"];
   ];
   gOut
  ];
```

## Functions defining the matrices and spectra (definitions are in Section 2.3 of arXiv:2010.10769 [math.CO] v1)

NOTE : For spectra and some related functions, there are two versions; the first outputs the spectrum only, the second (marked S for steps) shows the steps of the computation.

```
(* function incidenceMatrix takes one graph as input and
  outputs the incidence matrix as a SparseArray. Note that the built-
 in function IncidenceMatrix is similar but uses a different convention,
and in particular allows values other than +1 and -1 for loops. *)
incidenceMatrix[g0_] := Module[{graph = g0, verts, edges, i, j, m, n, inc},
   verts = VertexList[graph];
   m = Length[verts];
   edges = EdgeList[graph];
   n = Length[edges];
   inc = Table[0, {i, 1, m}, {j, 1, n}];
   i = 1;
   While[i ≤ m,
    j = 1;
    While[j ≤ n,
     If[edges[[j]][[1]] == edges[[j]][[2]], ,
      If[edges[[j]][[1]] == verts[[i]], inc[[i, j]] = 1];
      If[edges[[j]][[2]] == verts[[i]], inc[[i, j]] = -1],
      If[edges[[j]][[1]] == verts[[i]], inc[[i, j]] = 1];
      If[edges[[j]][[2]] == verts[[i]], inc[[i, j]] = -1]];
     j++];
    i++];
   inc
  ];

(* function laplaceSpec takes a graph as input and outputs the Laplace spectrum *)
laplaceSpec[g0_] := Module[{graph = g0, inc},
   inc = incidenceMatrix[graph];
   Eigenvalues[inc.Transpose[inc] ] ];
laplaceSpecS[g0_] := Module[{graph = g0, inc},
   inc = incidenceMatrix[graph];
   Print[graph];
   Print["Incidence matrix:"];
   Print[MatrixForm[inc]];
   Print["Laplacian:"];
   Print[MatrixForm[inc.Transpose[inc]]];
   Print["Laplace spectrum:"];
   Eigenvalues[inc.Transpose[inc] ] ];

(* function adjacencyMatrixBinary takes a
 graph as input and outputs the binary adjacency matrix,
i.e. the entry is 1 if there is an edge and 0 otherwise *)
```

```
adjacencyMatrixBinary[g0_] := Module[{graph = g0, inc, verts, n, i, j},
   verts = VertexList[graph];
   n = Length[verts];
   inc = Table[0, {i, 1, n}, {j, 1, n}];
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     If[MemberQ[ EdgeList[graph] , verts[[i]] ↔ verts[[j]] ],
      inc[[i, j]] = 1];
     j++];
    i++];
   inc] ;


(* function adjacencySpecBinary takes a graph
 as input and outputs the binary adjacency spectrum *)
adjacencySpecBinary[g0_] := Module[{graph = g0, inc},
   inc = adjacencyMatrixBinary[graph];
   Eigenvalues[inc ] ];
adjacencySpecBinaryS[g0_] := Module[{graph = g0, inc},
   inc = adjacencyMatrixBinary[graph];
   Print[graph];
   Print["Binary adjacency matrix:"];
   Print[MatrixForm[inc]];
   Print["Binary adjacency spectrum:"];
   Eigenvalues[inc ] ];

(* function adjacencySpecSymmetricBinary takes a graph as
 input and outputs the symmetric binary adjacency spectrum *)
adjacencySpecSymmetricBinary[g0_] := Module[{graph = g0, inc},
   inc = adjacencyMatrixBinary[graph];
   Eigenvalues[inc.Transpose[inc] ] ];
adjacencySpecSymmetricBinaryS[g0_] := Module[{graph = g0, inc},
   inc = adjacencyMatrixBinary[graph];
   Print[graph];
   Print["Binary adjacency matrix:"];
   Print[MatrixForm[inc]];
   Print["Binary adjacency matrix times its transpose:"];
   Print[MatrixForm[inc.Transpose[inc]]];
   Print["Symmetric binary adjacency spectrum:"];
   Eigenvalues[inc.Transpose[inc] ] ];


(* function adjacencyMatrix takes a graph as input and outputs the adjacency matrix,
i.e. the entry is k if there are k edges and 0 otherwise. *)
adjacencyMatrix[g0_] := Module[{graph = g0, inc, verts, n, i, j},
   verts = VertexList[graph];
   n = Length[verts];
   inc = Table[0, {i, 1, n}, {j, 1, n}];
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     inc[[i, j]] = Count[EdgeList[graph], verts[[i]] ↔ verts[[j]]];
```

```
      j++];
     i++];
    inc] ;


(* function adjacencySpec takes a graph
 as input and outputs the adjacency spectrum *)
adjacencySpec[g0_] := Module[{graph = g0, inc},
    inc = adjacencyMatrix[graph];
    Eigenvalues[inc ] ];
adjacencySpecS[g0_] := Module[{graph = g0, inc},
    inc = adjacencyMatrix[graph];
    Print[graph];
    Print["Adjacency matrix:"];
    Print[MatrixForm[inc]];
    Print["Adjacency spectrum:"];
    Eigenvalues[inc ] ];


(* function adjacencySpecSymmetric takes a graph
 as input and outputs the symmetric adjacency spectrum *)
adjacencySpecSymmetric[g0_] := Module[{graph = g0, inc},
    inc = adjacencyMatrix[graph];
    Eigenvalues[inc.Transpose[inc] ] ];
adjacencySpecSymmetricS[g0_] := Module[{graph = g0, inc},
    inc = adjacencyMatrix[graph];
    Print[graph];
    Print["Adjacency matrix:"];
    Print[MatrixForm[inc]];
    Print["Adjacency matrix times its transpose:"];
    Print[MatrixForm[inc.Transpose[inc]]];
    Print["Symmetric adjacency spectrum:"];
    Eigenvalues[inc.Transpose[inc] ] ];


(* function lineAdjacencyMatrix takes a graph
 as input and outputs the line adjacency matrix. *)
lineAdjacencyMatrix[g0_] := Module[{graph = g0, edges, n, inc, i, j},
    edges = EdgeList[graph];
    n = Length[edges];
    inc = Table[0, {i, 1, n}, {j, 1, n}];
    i = 1;
    While[i ≤ n,
     j = 1;
     While[j ≤ n,
      If[edges[[i]][[2]] == edges[[j]][[1]],
       inc[[i, j]] = 1];
      j++];
     i++];
    inc];


(* function lineGraph takes a graph as input and outputs
 the line graph (vertices are edges of the original graph,
  and there is an edge if the original edges are composable) *)
```

```
lineGraph[g0_] := Module[{graph = g0, inc, verts, edges, newEdges, n, i, j, gOut},
   edges = EdgeList[graph];
   n = Length[edges];
   verts = Table[e_{i,edges[[i]][[1]],edges[[i]][[2]]}, {i, 1, n}];
   newEdges = {};
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     If[edges[[i]][[2]] == edges[[j]][[1]],
      AppendTo[newEdges, e_{i,edges[[i]][[1]],edges[[i]][[2]]} ⟷ e_{j,edges[[j]][[1]],edges[[j]][[2]]}]];
     j++];
    i++];
   (* newEdges=DeleteDuplicates[newEdges]; *)
   gOut = Graph[verts, newEdges, VertexLabels → "Name", ImageSize → 600];
   gOut
  ];


(* function lineAdjacencySpec takes a graph as input and outputs the line-
 adjacency spectrum *)
lineAdjacencySpec[g0_] := Module[{graph = g0, inc},
   inc = lineAdjacencyMatrix[graph];
   Eigenvalues[inc ] ];
lineAdjacencySpecS[g0_] := Module[{graph = g0, inc},
   inc = lineAdjacencyMatrix[graph];
   Print[graph];
   Print["Line graph:"];
   Print[lineGraph[graph]];
   Print["Line adjacency matrix:"];
   Print[MatrixForm[inc]];
   Print["Line adjacency spectrum:"];
   Eigenvalues[inc ] ];


(* function hermitianAdjacencyMatrix takes a graph
 as input and outputs the Hermitian adjacency matrix *)
hermitianAdjacencyMatrix[g0_] := Module[{graph = g0, inc, verts, n, i, j},
   verts = VertexList[graph];
   n = Length[verts];
   inc = Table[0, {i, 1, n}, {j, 1, n}];
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     If[MemberQ[ EdgeList[graph] , verts[[i]] ↔ verts[[j]] ],
      If[MemberQ[ EdgeList[graph] , verts[[j]] ↔ verts[[i]] ],
       inc[[i, j]] = 1,
       inc[[i, j]] = I],
      If[MemberQ[ EdgeList[graph] , verts[[j]] ↔ verts[[i]] ],
       inc[[i, j]] = -I,
       inc[[i, j]] = 0]];
     j++];
    i++];
   inc] ;
```

```
(* function hermitianSpectrum takes a graph as
 input and outputs the Hermitian adjacency spectrum *)
hermitianAdjacencySpectrum[g0_] := Module[{graph = g0, inc},
   inc = hermitianAdjacencyMatrix[graph];
   Eigenvalues[inc ] ];
hermitianAdjacencySpectrumS[g0_] := Module[{graph = g0, inc},
   inc = hermitianAdjacencyMatrix[graph];
   Print[graph];
   Print["Hermitian adjacency matrix:"];
   Print[MatrixForm[inc]];
   Print["Hermitian adjacency spectrum:"];
   Eigenvalues[inc ] ];

(* function skewAdjacencyMatrixBinary takes a graph
 as input and outputs the binary skew adjacency matrix *)
skewAdjacencyMatrixBinary[g0_] :=
  adjacencyMatrixBinary[g0] - Transpose[adjacencyMatrixBinary[g0]];


(* function skewAdjacencySpecBinary takes a graph
 as input and outputs the binary skew adjacency spectrum *)
skewAdjacencySpecBinary[g0_] := Module[{graph = g0, inc},
   inc = skewAdjacencyMatrixBinary[graph];
   Eigenvalues[inc] ];
skewAdjacencySpecBinaryS[g0_] := Module[{graph = g0, inc},
   inc = skewAdjacencyMatrixBinary[graph];
   Print[graph];
   Print["Binary adjacency matrix:"];
   Print[MatrixForm[adjacencyMatrixBinary[graph]]];
   Print["Binary skew adjacency matrix:"];
   Print[MatrixForm[inc]];
   Print["Binary skew adjacency spectrum:"];
   Eigenvalues[inc] ];


(* function skewAdjacencyMatrix takes a graph
 as input and outputs the skew adjacency matrix *)
skewAdjacencyMatrix[g0_] := adjacencyMatrix[g0] - Transpose[adjacencyMatrix[g0]];


(* function skewAdjacencySpec takes a graph
 as input and outputs the skew adjacency spectrum *)
skewAdjacencySpec[g0_] := Module[{graph = g0, inc},
   inc = skewAdjacencyMatrix[graph];
   Eigenvalues[inc] ];
skewAdjacencySpecS[g0_] := Module[{graph = g0, inc},
   inc = skewAdjacencyMatrix[graph];
   Print[graph];
   Print["Adjacency matrix:"];
   Print[MatrixForm[adjacencyMatrix[graph]]];
   Print["Skew adjacency matrix:"];
   Print[MatrixForm[inc]];
   Print["Skew adjacency spectrum:"];
   Eigenvalues[inc] ];
```

```
(* function skewLaplacianMatrix takes a graph
 as input and outputs the skew Laplacian matrix *)
skewLaplacianMatrix[g0_] := Module[{graph = g0, inc, verts, n, i, j, d},
   verts = VertexList[graph];
   n = Length[verts];
   d = Table[0, {i, 1, n}, {j, 1, n}];
   (* we compute in-degree - out-degree,
   but COUNTING EDGES, not vertices, so counting parallel edges. *)
   i = 1;
   While[i ≤ n,
    d[[i, i]] =
     Count[EdgeList[graph], verts[[i]] ↔ _] - Count[EdgeList[graph], _ ↔ verts[[i]]];
    i++];
   d - skewAdjacencyMatrix[graph]
  ];


(* function skewLaplaceSpec takes a graph
 as input and outputs the skew Laplacian spectrum *)
skewLaplaceSpec[g0_] := Module[{graph = g0},
   Eigenvalues[skewLaplacianMatrix[graph]]
  ];
skewLaplaceSpecS[g0_] := Module[{graph = g0},
   Print[graph];
   Print["Skew Laplacian matrix"];
   Print[MatrixForm[skewLaplacianMatrix[graph]]];
   Print["Skew Laplace spectrum"];
   Eigenvalues[skewLaplacianMatrix[graph]]
  ];


(* function skewLaplacianMatrixBinary takes a graph
 as input and outputs the binary skew Laplacian matrix *)
skewLaplacianMatrixBinary[g0_] := Module[{graph = g0, inc, verts, n, i, j, d},
   verts = VertexList[graph];
   n = Length[verts];
   d = Table[0, {i, 1, n}, {j, 1, n}];
   i = 1;
   (* we compute in-degree - out-degree,
   but COUNTING VERTICES, not edges, so ignoring parallel edges. *)
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     If[MemberQ[EdgeList[graph], verts[[i]] ↔ verts[[j]] ],
      d[[i, i]]++];
     If[MemberQ[EdgeList[graph], verts[[j]] ↔ verts[[i]] ],
      d[[i, i]]--];
     j++];
    i++];
   d - skewAdjacencyMatrixBinary[graph]
  ];
```

```
(* function skewLaplaceSpecBinary takes a graph as
 input and outputs the binary skew Laplacian spectrum *)
skewLaplaceSpecBinary[g0_] := Module[{graph = g0},
   Eigenvalues[skewLaplacianMatrixBinary[graph]]
  ];
skewLaplaceSpecBinaryS[g0_] := Module[{graph = g0},
   Print[graph];
   Print["Binary skew Laplacian matrix"];
   Print[MatrixForm[skewLaplacianMatrixBinary[graph]]];
   Print["Binary skew Laplace spectrum"];
   Eigenvalues[skewLaplacianMatrixBinary[graph]]
  ];


(* function transitionProbabilityMatrixBinary takes a graph
 as input and outputs the binary transition probability matrix *)
transitionProbabilityMatrixBinary[g0_] := Module[{graph = g0, inc, verts, n, i, j, d},
   verts = VertexList[graph];
   n = Length[verts];
   inc = Table[0, {i, 1, n}, {j, 1, n}];
   d = Table[0, {i, 1, n}];
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     If[MemberQ[ EdgeList[graph] , verts[[i]] ⟷ verts[[j]]],
      d[[i]] ++];
     j++];
    i++];
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     If[MemberQ[ EdgeList[graph] ,  verts[[i]] ⟷ verts[[j]] ],
      inc[[i, j]] = 1 / d[[i]]];
     j++];
    i++];
   inc] ;


(* function normalizedLaplacianBinary takes a graph as
 input and outputs the binary normalized Laplacian matrix *)
normalizedLaplacianBinary[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
   verts = VertexList[graph];
   n = Length[verts];
   p = transitionProbabilityMatrixBinary[graph];
    (* define phi, the eigenvector with  *)
   eig = Eigensystem[Transpose[p] ];
   i = 1;
   While[i ≤ n,
    If[
     AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
     phi = eig[[2, i]];
```

```
      i = n + 1];
     i++];
    Clear[i];
    (* normalize phi *)
    phiSum = Sum[phi[[i]], {i, 1, n}];
    phi = phi / phiSum;
    (* Phi is the diagonal matrix with entries of phi *)
    Phi = DiagonalMatrix[phi];
    IdentityMatrix[n] - (Sqrt[Phi].p.Sqrt[Inverse[Phi]] +
        Sqrt[Inverse[Phi]].ConjugateTranspose[p].Sqrt[Phi]) / 2
   ];
normalizedLaplacianBinaryS[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
    verts = VertexList[graph];
    n = Length[verts];
    p = transitionProbabilityMatrixBinary[graph];
    Print["Binary transition probability matrix:"];
    Print[MatrixForm[p]];
    (* define phi, the eigenvector with  *)
    eig = Eigensystem[Transpose[p] ];
    i = 1;
    While[i ≤ n,
     If[
      AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
      phi = eig[[2, i]];
      i = n + 1];
     i++];
    Clear[i];
    (* normalize phi *)
    phiSum = Sum[phi[[i]], {i, 1, n}];
    phi = phi / phiSum;
    Print["Perron-Frobenius vector:"];
    Print[phi];
    (* Phi is the diagonal matrix with entries of phi *)
    Phi = DiagonalMatrix[phi];
    Print["Perron-Frobenius vector as a matrix:"];
    Print[MatrixForm[Phi]];
    IdentityMatrix[n] - (Sqrt[Phi].p.Sqrt[Inverse[Phi]] +
        Sqrt[Inverse[Phi]].ConjugateTranspose[p].Sqrt[Phi]) / 2
   ];


(* function combinatorialLaplacianBinary takes a graph as
 input and outputs the binary combinatorial Laplacian matrix *)
combinatorialLaplacianBinary[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
    verts = VertexList[graph];
    n = Length[verts];
    p = transitionProbabilityMatrixBinary[graph];
    (* define phi, the eigenvector with  *)
    eig = Eigensystem[Transpose[p] ];
    i = 1;
    While[i ≤ n,
     If[
```

```
      AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
      phi = eig[[2, i]];
      i = n + 1];
     i++];
    (* normalize phi *)
    phiSum = Sum[phi[[i]], {i, 1, n}];
    phi = phi / phiSum;
    (* Phi is the diagonal matrix with entries of phi *)
    Phi = DiagonalMatrix[phi];
    Phi - (Phi.p + ConjugateTranspose[p].Phi) / 2
   ];
combinatorialLaplacianBinaryS[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
    verts = VertexList[graph];
    n = Length[verts];
    p = transitionProbabilityMatrixBinary[graph];
    Print["Binary transition probability matrix:"];
    Print[MatrixForm[p]];
    (* define phi, the eigenvector with  *)
    eig = Eigensystem[Transpose[p] ];
    i = 1;
    While[i ≤ n,
     If[
      AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
      phi = eig[[2, i]];
      i = n + 1];
     i++];
    (* normalize phi *)
    phiSum = Sum[phi[[i]], {i, 1, n}];
    phi = phi / phiSum;
    Print["Perron-Frobenius vector:"];
    Print[phi];
    (* Phi is the diagonal matrix with entries of phi *)
    Phi = DiagonalMatrix[phi];
    Print["Perron-Frobenius vector as a matrix:"];
    Print[MatrixForm[Phi]];
    Phi - (Phi.p + ConjugateTranspose[p].Phi) / 2
   ];


(* function normalizedLaplaceSpecBinary takes a graph as
 input and outputs the binary normalized Laplacian spectrum *)
normalizedLaplaceSpecBinary[g0_] := Module[{graph = g0, inc},
    inc = normalizedLaplacianBinary[graph];
    Eigenvalues[inc ] ];
normalizedLaplaceSpecBinaryS[g0_] := Module[{graph = g0, inc},
    Print[graph];
    inc = normalizedLaplacianBinaryS[graph];
    Print["Binary normalized Laplacian:"];
    Print[MatrixForm[inc]];
    Print["Binary normalized Laplace spectrum:"];
    Eigenvalues[inc ] ];


(* function combinatorialLaplaceSpecBinary takes a graph as
```

```
 input and outputs the binary combinatorial Laplacian spectrum *)
combinatorialLaplaceSpecBinary[g0_] := Module[{graph = g0, inc},
   inc = combinatorialLaplacianBinary[graph];
   Eigenvalues[inc ] ];
combinatorialLaplaceSpecBinaryS[g0_] := Module[{graph = g0, inc},
   Print[graph];
   inc = combinatorialLaplacianBinaryS[graph];
   Print["Binary combinatorial Laplacian:"];
   Print[MatrixForm[inc]];
   Print["Binary combinatorial Laplace spectrum:"];
   Eigenvalues[inc ] ];


(* function transitionProbabilityMatrix takes a graph
 as input and outputs the transition probability matrix *)
transitionProbabilityMatrix[g0_] := Module[{graph = g0, inc, verts, edges, n, i, j, d},
   verts = VertexList[graph];
   edges = EdgeList[graph];
   n = Length[verts];
   inc = Table[0, {i, 1, n}, {j, 1, n}];
   i = 1;
   While[i ≤ n,
    j = 1;
    While[j ≤ n,
     d = Count[edges, verts[[i]] ↔ _ ];
     If[d > 0,
      inc[[i, j]] = Count[edges, verts[[i]] ↔ verts[[j]]] / d];
     j++];
    i++];
   inc] ;


(* function normalizedLaplacian takes a graph
 as input and outputs the normalized Laplacian matrix *)
normalizedLaplacian[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
   verts = VertexList[graph];
   n = Length[verts];
   p = transitionProbabilityMatrix[graph];
   (* define phi, the eigenvector with  *)
   eig = Eigensystem[Transpose[p] ];
   i = 1;
   While[i ≤ n,
    If[
     AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
     phi = eig[[2, i]];
     i = n + 1];
    i++];
   Clear[i];
   (* normalize phi *)
   phiSum = Sum[phi[[i]], {i, 1, n}];
   phi = phi / phiSum;
   (* Phi is the diagonal matrix with entries of phi *)
   Phi = DiagonalMatrix[phi];
```

```
    IdentityMatrix[n] - (Sqrt[Phi].p.Sqrt[Inverse[Phi]] +
        Sqrt[Inverse[Phi]].ConjugateTranspose[p].Sqrt[Phi]) / 2
   ];
normalizedLaplacianS[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
   verts = VertexList[graph];
   n = Length[verts];
   p = transitionProbabilityMatrix[graph];
   Print["Transition probability matrix:"];
   Print[MatrixForm[p]];
   (* define phi, the eigenvector with  *)
   eig = Eigensystem[Transpose[p] ];
   i = 1;
   While[i ≤ n,
    If[
     AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
     phi = eig[[2, i]];
     i = n + 1];
    i++];
   Clear[i];
   (* normalize phi *)
   phiSum = Sum[phi[[i]], {i, 1, n}];
   phi = phi / phiSum;
   Print["Perron-Frobenius vector:"];
   Print[phi];
   (* Phi is the diagonal matrix with entries of phi *)
   Phi = DiagonalMatrix[phi];
   Print["Perron-Frobenius vector as a matrix:"];
   Print[MatrixForm[Phi]];
   IdentityMatrix[n] - (Sqrt[Phi].p.Sqrt[Inverse[Phi]] +
        Sqrt[Inverse[Phi]].ConjugateTranspose[p].Sqrt[Phi]) / 2
   ];


(* function combinatorialLaplacian takes a graph
 as input and outputs the combinatorial Laplacian matrix *)
combinatorialLaplacian[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
   verts = VertexList[graph];
   n = Length[verts];
   p = transitionProbabilityMatrix[graph];
   (* define phi, the eigenvector with  *)
   eig = Eigensystem[Transpose[p] ];
   i = 1;
   While[i ≤ n,
    If[
     AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
     phi = eig[[2, i]];
     i = n + 1];
    i++];
   (* normalize phi *)
   phiSum = Sum[phi[[i]], {i, 1, n}];
   phi = phi / phiSum;
   Phi = DiagonalMatrix[phi];
```

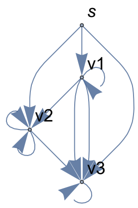
```
    Phi - (Phi.p + ConjugateTranspose[p].Phi) / 2
   ];
combinatorialLaplacianS[g0_] :=
  Module[{graph = g0, inc, verts, n, i, j, phi, p, eig, Phi, phiSum},
    verts = VertexList[graph];
    n = Length[verts];
    p = transitionProbabilityMatrix[graph];
    Print["Transition probability matrix:"];
    Print[MatrixForm[p]];
    (* define phi, the eigenvector with  *)
    eig = Eigensystem[Transpose[p] ];
    i = 1;
    While[i ≤ n,
     If[
      AllTrue[Table[j, {j, 1, n}], eig[[2, i]][[#]] > 0 &],
      phi = eig[[2, i]];
      i = n + 1];
     i++];
    (* normalize phi *)
    phiSum = Sum[phi[[i]], {i, 1, n}];
    phi = phi / phiSum;
    Print["Perron-Frobenius vector:"];
    Print[phi];
    Phi = DiagonalMatrix[phi];
    Print["Perron-Frobenius vector as a matrix:"];
    Print[MatrixForm[Phi]];
    Phi - (Phi.p + ConjugateTranspose[p].Phi) / 2
   ];


(* function normalizedLaplaceSpec takes a graph
 as input and outputs the normalized Laplacian spectrum *)
normalizedLaplaceSpec[g0_] := Module[{graph = g0, inc},
    inc = normalizedLaplacian[graph];
    Eigenvalues[inc ] ];
normalizedLaplaceSpecS[g0_] := Module[{graph = g0, inc},
    Print[graph];
    inc = normalizedLaplacianS[graph];
    Print["Normalized Laplacian:"];
    Print[MatrixForm[inc]];
    Print["Normalized Laplace spectrum:"];
    Eigenvalues[inc ] ];

(* function combinatorialLaplaceSpec takes a graph
 as input and outputs the combinatorial Laplacian spectrum *)
combinatorialLaplaceSpec[g0_] := Module[{graph = g0, inc},
    inc = combinatorialLaplacianS[graph];
    Eigenvalues[inc ] ];
combinatorialLaplaceSpecS[g0_] := Module[{graph = g0, inc},
    Print[graph];
    inc = combinatorialLaplacianS[graph];
    Print["Combinatorial Laplacian:"];
    Print[MatrixForm[inc]];
    Print["Combinatorial Laplace spectrum:"];
    Eigenvalues[inc ] ];
```

# Example 5.1 (of arXiv:2010.10769 [math.CO] v1)

## Definitions of the graphs

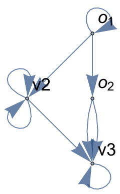
```
(* Define the graph D1 *)
Print["Graph D1"];
d1 = Graph[{v1, v2, v3}, {v1 ⟶ v1, v1 ⟶ v2, v2 ⟶ v2,
   v2 ⟶ v2, v1 ⟶ v3, v1 ⟶ v3, v2 ⟶ v3, v3 ⟶ v3}, VertexLabels → "Name"]

(* Define the graph D1^(S) *)
(* computed using d1S=moveSinv[d1] *)
Print["Graph D1^(S)"];
```

d1S =



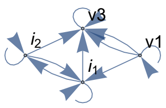

```
(* Define the graph D1^(R) *)
Print["Graph D1^(R)"];
d1R = Graph[{v1, v2, v3, r}, {v1 ⟶ v1, v1 ⟶ v2, v2 ⟶ v2, v2 ⟶ r,
   r ⟶ v2, v1 ⟶ v3, v1 ⟶ v3, v2 ⟶ v3, v3 ⟶ v3}, VertexLabels → "Name"]

(* Define the graph D1^(O) *)
(* computed using d1O=moveO[d1,o] *)
Print["Graph D1^(O)"];
```

d1O =



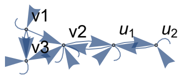

```
(* Define the graph D1^(I) *)
Print["Graph D1^(I)"];
(* computed using d1I=moveI[d1,i] *)
```

d1I =

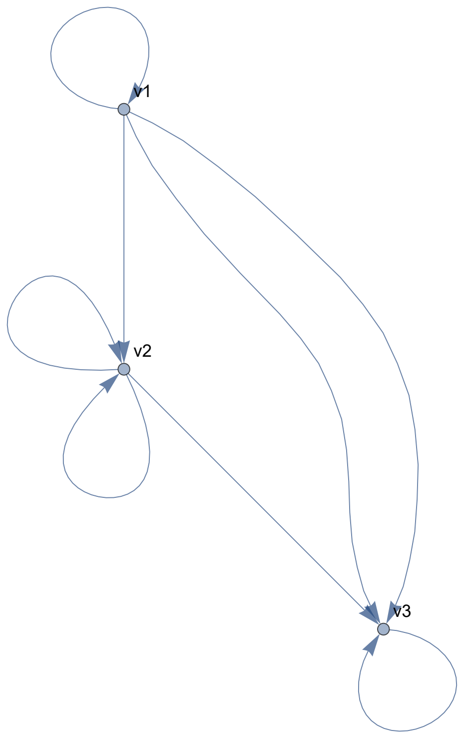



```
(* Define the graph D1^(C) *)
Print["Graph D1^(C)"];
(* computed using d1C=moveC[d1,u] *)
```

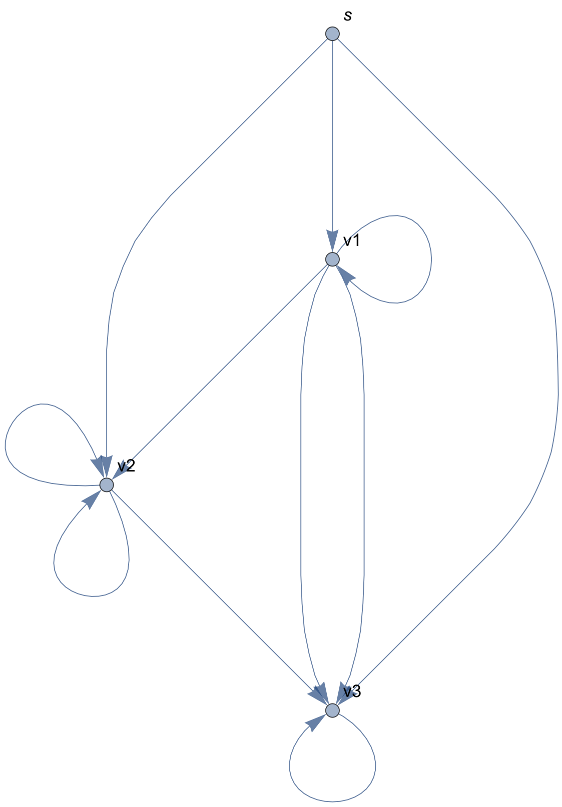
d1C =



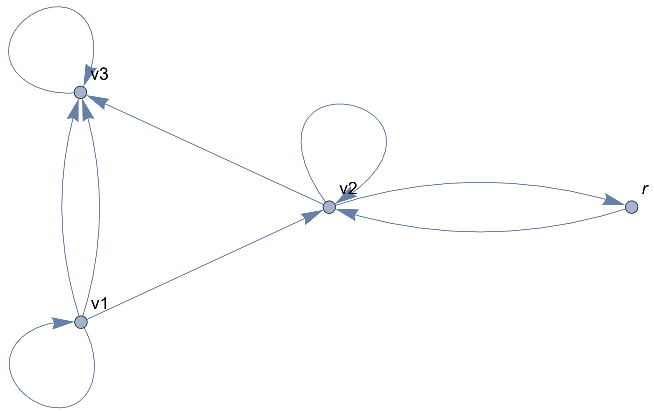

Graph $D_1$



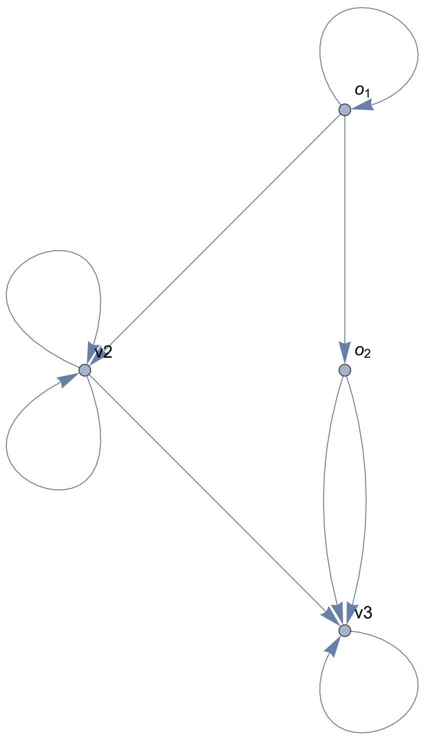

Graph $D_1^{(S)}$



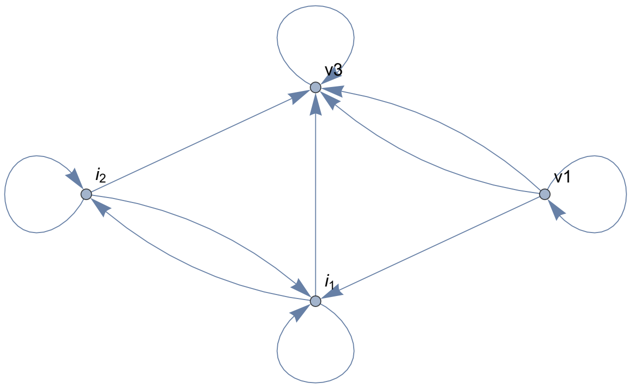

Graph $D_1^{(R)}$



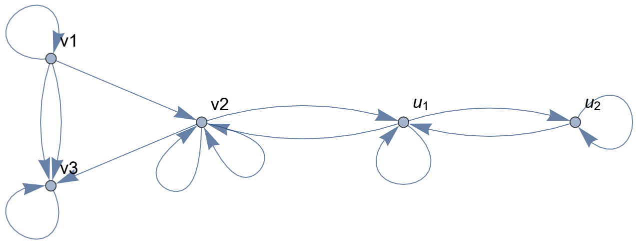

Graph $D_1^{(0)}$




Graph $D_1^{(I)}$




Graph $D_1^{(C)}$



## Laplace spectra

```
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Laplace spectra"];
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1"];
laplaceSpecS[d1]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(S)"];
laplaceSpecS[d1S]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(R)"];
laplaceSpecS[d1R]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(O)"];
laplaceSpecS[d1O]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(I)"];
laplaceSpecS[d1I]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(C)"];
laplaceSpecS[d1C]
% // N
```

----------------------------------------------------------------------

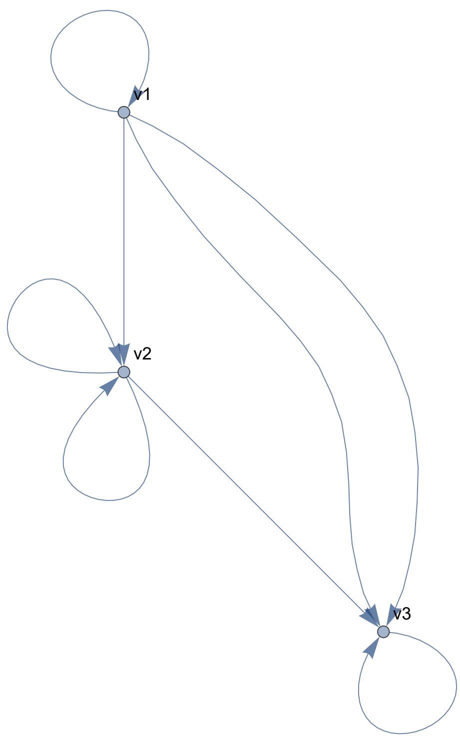
----------------------------------------------------------------------

Laplace spectra

----------------------------------------------------------------------

----------------------------------------------------------------------

Graph $D_1$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$$

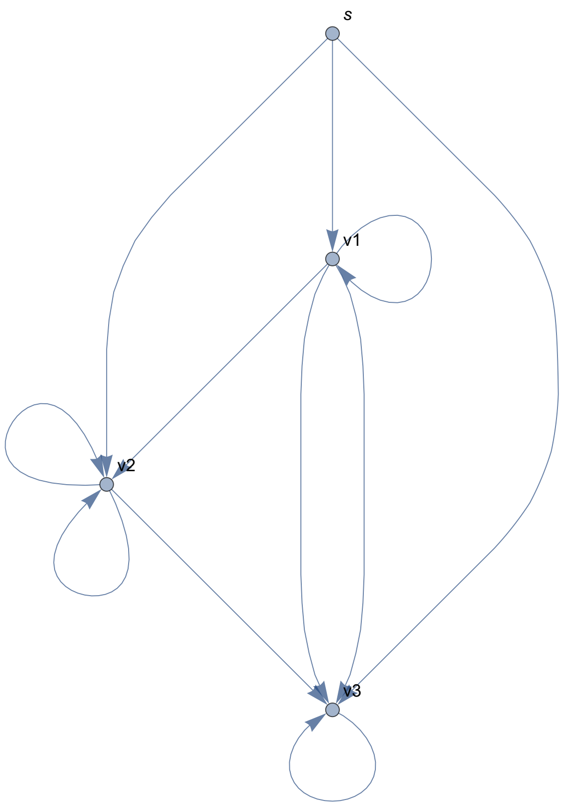
Laplace spectrum:

{5, 3, 0}

{5., 3., 0.}

------------------------------------------------------------------------

Graph $D_1^{(S)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

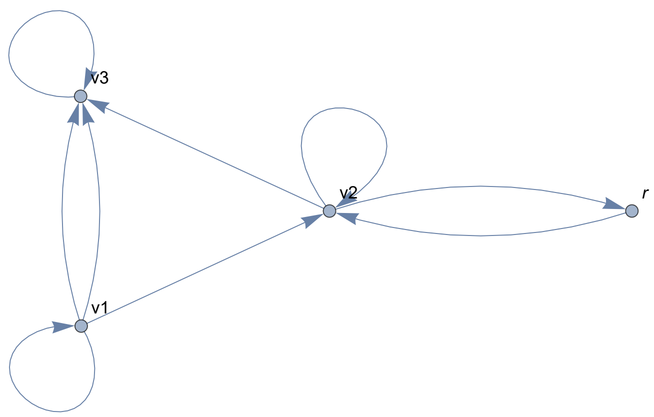
Laplace spectrum:

{6, 4, 4, 0}

{6., 4., 4., 0.}

------------------------------------------------------------------------

Graph $D_1^{(R)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

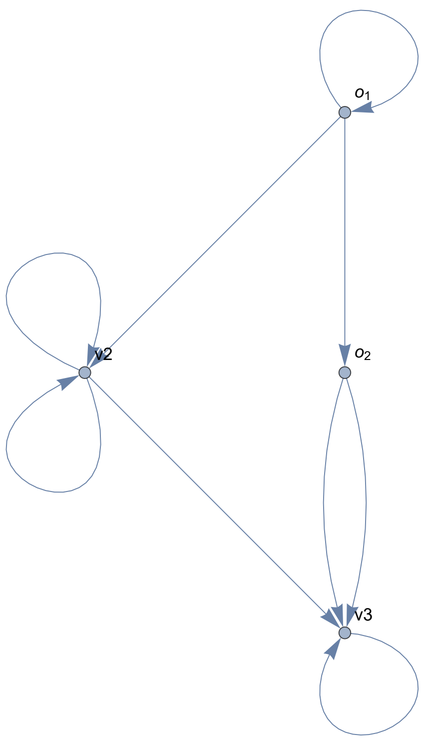
Laplace spectrum:

$$\left\{\frac{1}{2}\left(7+\sqrt{17}\right), 5, \frac{1}{2}\left(7-\sqrt{17}\right), 0\right\}$$

{5.56155, 5., 1.43845, 0.}

-----------------------------------------------------------------------

Graph $D_1^{(0)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

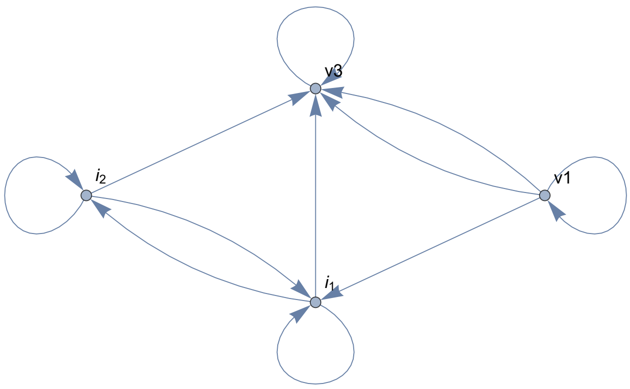
Laplace spectrum:

$\{4+\sqrt{2}, 4-\sqrt{2}, 2, 0\}$

{5.41421, 2.58579, 2., 0.}

----------------------------------------------------------------------

Graph $D_1^{(I)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

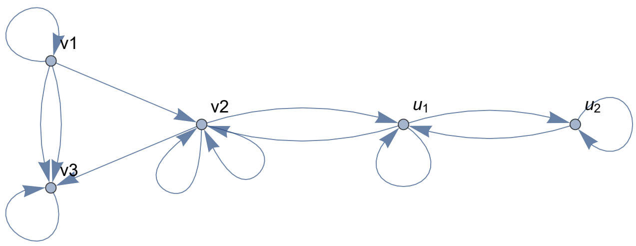
Laplace spectrum:

$\{6, 4+\sqrt{2}, 4-\sqrt{2}, 0\}$

{6., 5.41421, 2.58579, 0.}

----------------------------------------------------------------------

Graph $D_1^{(C)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Laplace spectrum:

$\{\text{Root}[-20+32\,\#1-11\,\#1^2+\#1^3\,\&, 3], 5,$
$\text{Root}[-20+32\,\#1-11\,\#1^2+\#1^3\,\&, 2], \text{Root}[-20+32\,\#1-11\,\#1^2+\#1^3\,\&, 1], 0\}$

{6.6262, 5., 3.51514, 0.858664, 0.}

## Adjacency spectra

```
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Adjacency spectra"];
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Graph D_1"];
adjacencySpecS[d1]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D_1^(S)"];
adjacencySpecS[d1S]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D_1^(O)"];
adjacencySpecS[d1O]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D_1^(I)"];
adjacencySpecS[d1I]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D_1^(R)"];
adjacencySpecS[d1R]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D_1^(C)"];
adjacencySpecS[d1C]
% // N
```

------------------------------------------------------------------------

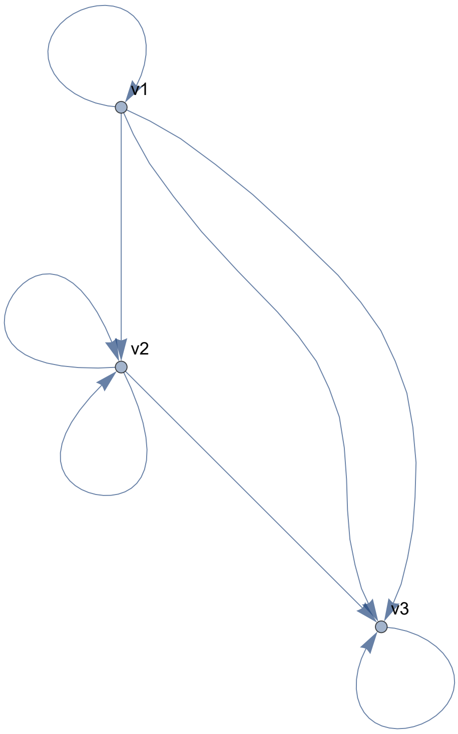
------------------------------------------------------------------------

Adjacency spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_1$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

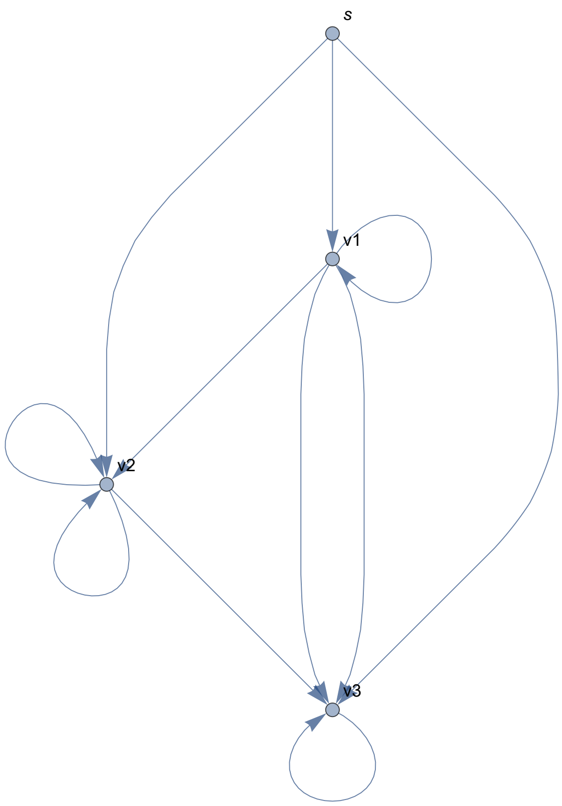
Adjacency spectrum:

{2, 1, 1}

{2., 1., 1.}

-----------------------------------------------------------------------

Graph $D_1^{(S)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

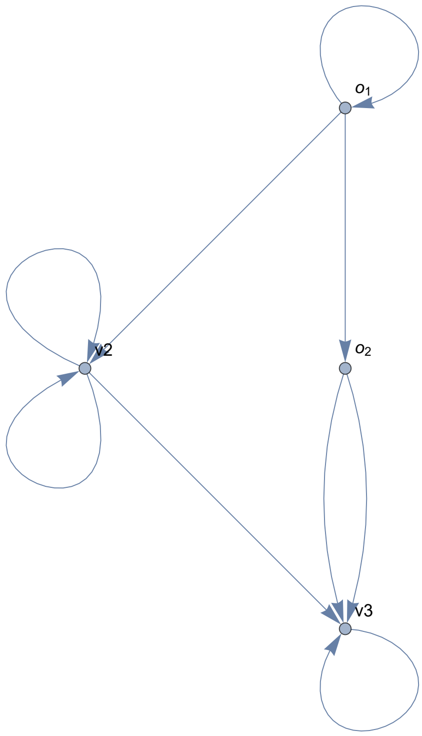
Adjacency spectrum:

{2, 1, 1, 0}

{2., 1., 1., 0.}

------------------------------------------------------------------------

Graph $D_1^{(0)}$




Adjacency matrix:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

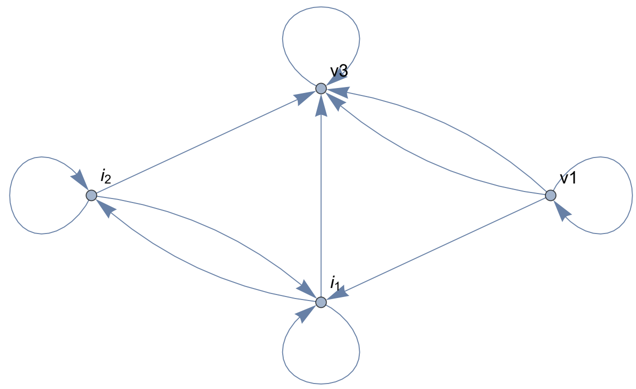
Adjacency spectrum:

{2, 1, 1, 0}

{2., 1., 1., 0.}

---

Graph $D_1^{(I)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

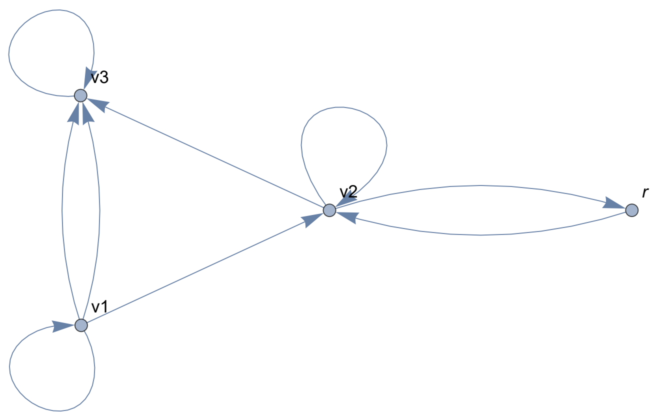
Adjacency spectrum:

{2, 1, 1, 0}

{2., 1., 1., 0.}

---

Graph $D_1^{(R)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

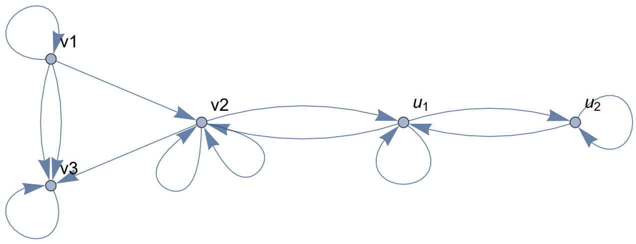
Adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), 1, 1, \frac{1}{2}\left(1-\sqrt{5}\right)\right\}$$

{1.61803, 1., 1., -0.618034}

---

Graph $D_1^{(C)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

Adjacency spectrum:

{Root[1 + 3 #1 - 4 #1^2 + #1^3 &, 3],
Root[1 + 3 #1 - 4 #1^2 + #1^3 &, 2], 1, 1, Root[1 + 3 #1 - 4 #1^2 + #1^3 &, 1]}

{2.80194, 1.44504, 1., 1., -0.24698}

## Binary adjacency spectra

```
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Binary adjacency spectra"];
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1"];
adjacencySpecBinaryS[d1]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(S)"];
adjacencySpecBinaryS[d1S]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(O)"];
adjacencySpecBinaryS[d1O]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(R)"];
adjacencySpecBinaryS[d1R]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(I)"];
adjacencySpecBinaryS[d1I]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_1^(C)"];
adjacencySpecBinaryS[d1C]
% // N
```

--------------------------------------------------------------------------------

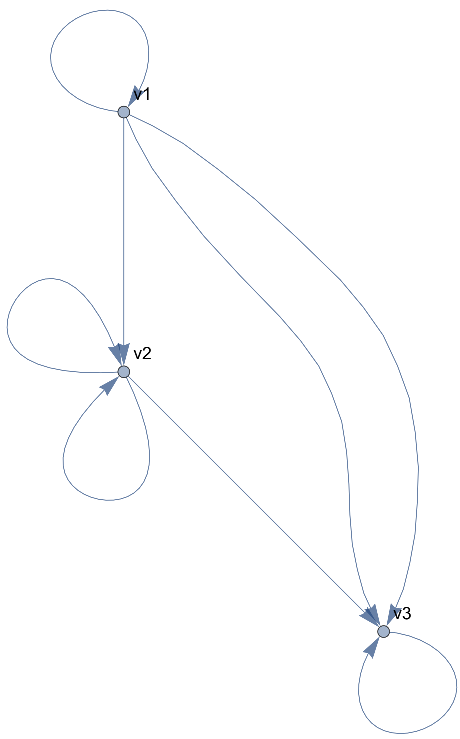
--------------------------------------------------------------------------------

Binary adjacency spectra

--------------------------------------------------------------------------------

--------------------------------------------------------------------------------

Graph $D_1$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

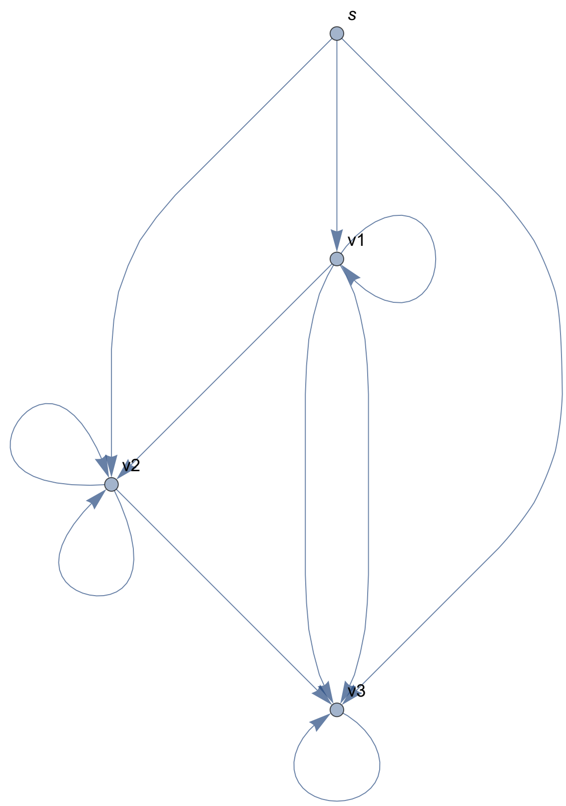
Binary adjacency spectrum:

{1, 1, 1}

{1., 1., 1.}

---------------------------------------------------------------------

Graph $D_1^{(S)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

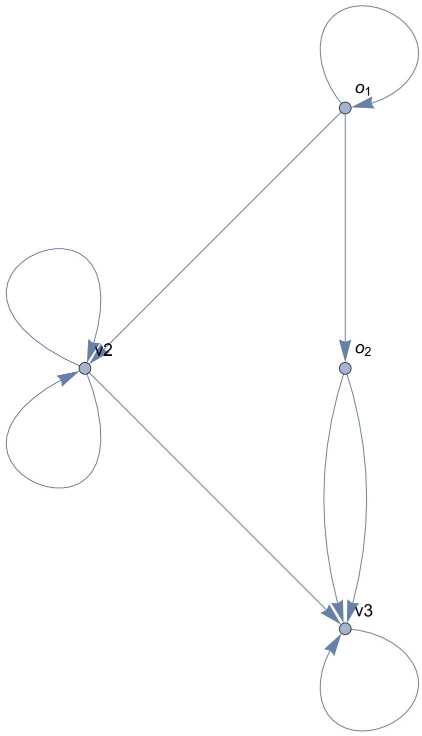
Binary adjacency spectrum:

{1, 1, 1, 0}

{1., 1., 1., 0.}

-----------------------------------------------------------------------

Graph $D_1^{(0)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

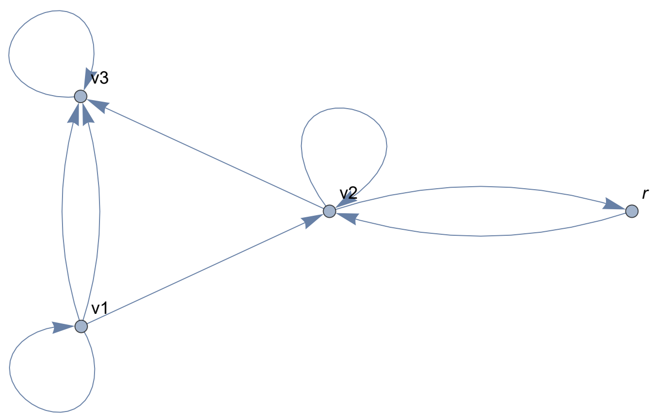
Binary adjacency spectrum:

{1, 1, 1, 0}

{1., 1., 1., 0.}

---

Graph $D_1^{(R)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

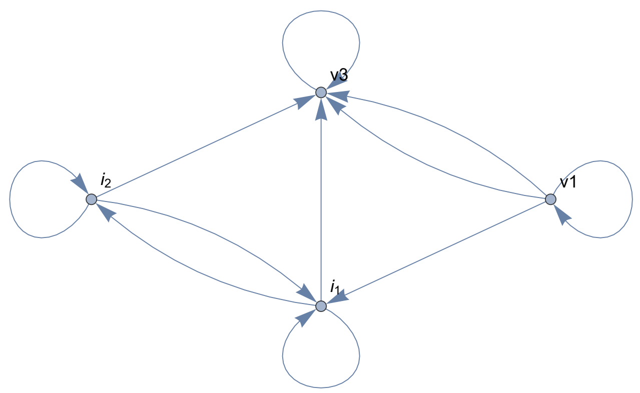
Binary adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), 1, 1, \frac{1}{2}\left(1-\sqrt{5}\right)\right\}$$

{1.61803, 1., 1., -0.618034}

---

Graph $D_1^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

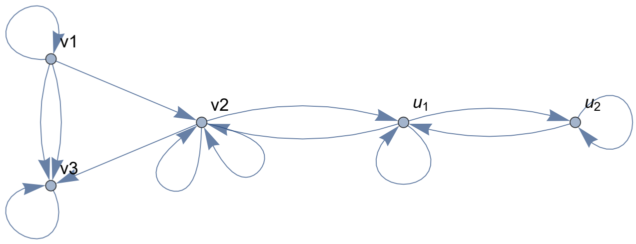
Binary adjacency spectrum:

{2, 1, 1, 0}

{2., 1., 1., 0.}

--------------------------------------------------------------------------------

Graph $D_1^{(C)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Binary adjacency spectrum:

$\{1 + \sqrt{2}, 1, 1, 1, 1 - \sqrt{2}\}$

$\{2.41421, 1., 1., 1., -0.414214\}$

## Symmetric adjacency spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Symmetric adjacency spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1"];
adjacencySpecSymmetricS[d1]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(S)"];
adjacencySpecSymmetricS[d1S]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(R)"];
adjacencySpecSymmetricS[d1R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(O)"];
adjacencySpecSymmetricS[d1O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(I)"];
adjacencySpecSymmetricS[d1I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(C)"];
adjacencySpecSymmetricS[d1C]
% // N
```

------------------------------------------------------------------------

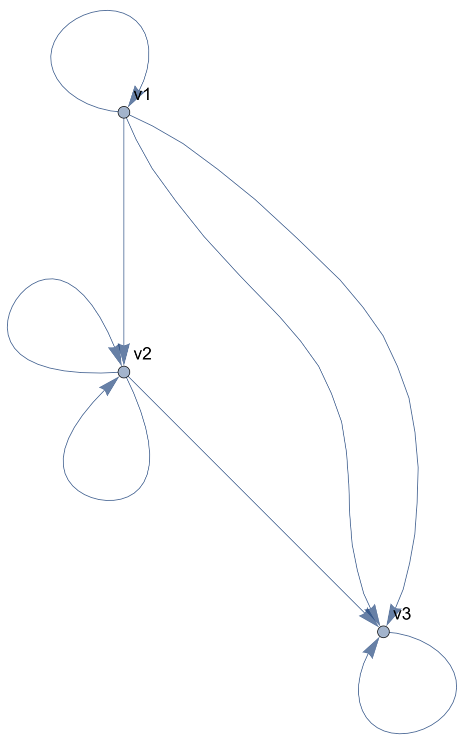
------------------------------------------------------------------------

Symmetric adjacency spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_1$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 6 & 4 & 2 \\ 4 & 5 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

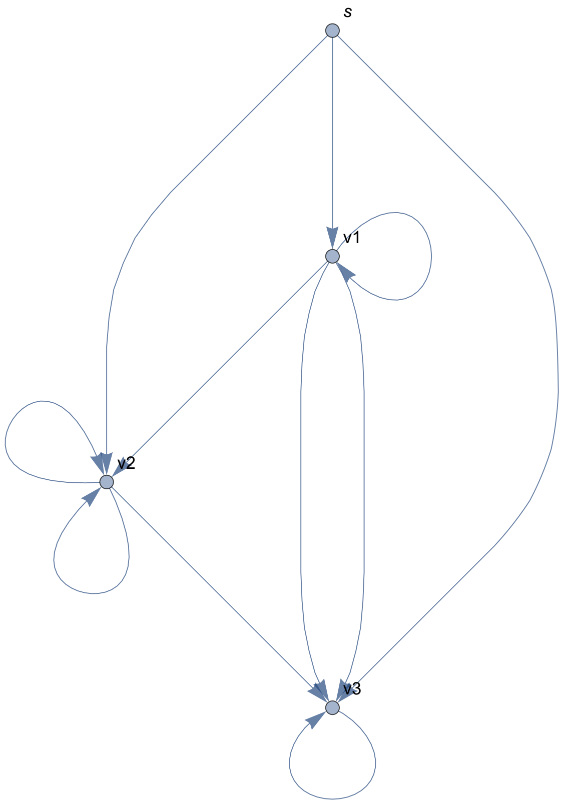
Symmetric adjacency spectrum:

$\{\text{Root}[-4 + 20\,\sharp1 - 12\,\sharp1^2 + \sharp1^3\ \&,\ 3],$
$\text{Root}[-4 + 20\,\sharp1 - 12\,\sharp1^2 + \sharp1^3\ \&,\ 2],\ \text{Root}[-4 + 20\,\sharp1 - 12\,\sharp1^2 + \sharp1^3\ \&,\ 1]\}$

{10.0494, 1.719, 0.231548}

------------------------------------------------------------------------

Graph $D_1^{(S)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 6 & 4 & 2 & 4 \\ 4 & 5 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & 3 & 1 & 3 \end{pmatrix}$$

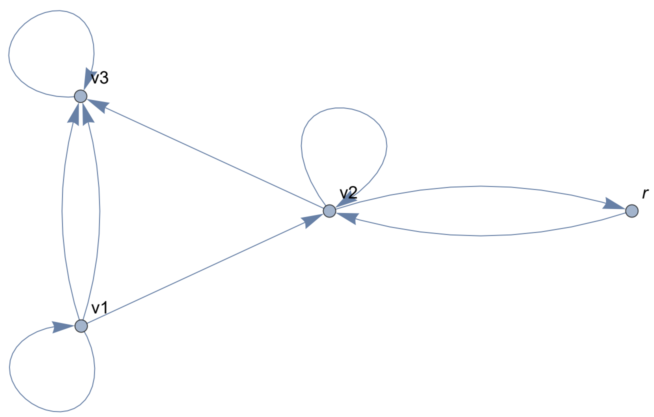
Symmetric adjacency spectrum:

{Root[-12 + 30 #1 - 15 #1^2 + #1^3 &, 3],
Root[-12 + 30 #1 - 15 #1^2 + #1^3 &, 2], Root[-12 + 30 #1 - 15 #1^2 + #1^3 &, 1], 0}

{12.7148, 1.74411, 0.541128, 0.}

------------------------------------------------------------------------

Graph $D_1^{(R)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

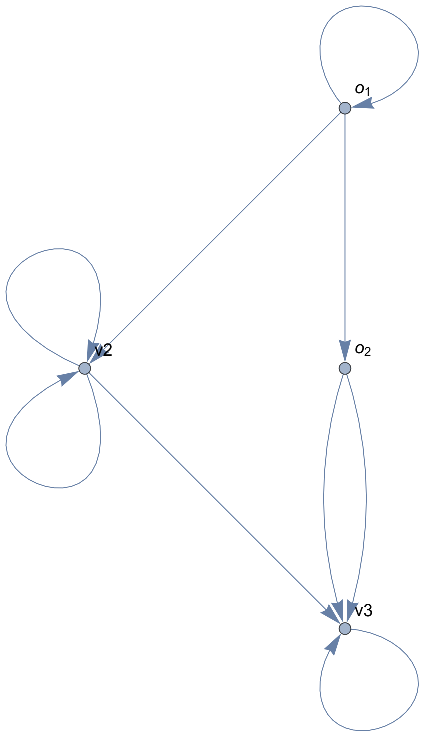
Symmetric adjacency spectrum:

$$\{\text{Root}[1 - 11\,\#1 + 21\,\#1^2 - 11\,\#1^3 + \#1^4\,\&, 4], \text{Root}[1 - 11\,\#1 + 21\,\#1^2 - 11\,\#1^3 + \#1^4\,\&, 3], \text{Root}[1 - 11\,\#1 + 21\,\#1^2 - 11\,\#1^3 + \#1^4\,\&, 2], \text{Root}[1 - 11\,\#1 + 21\,\#1^2 - 11\,\#1^3 + \#1^4\,\&, 1]\}$$

{8.73968, 1.46182, 0.684079, 0.114421}

---------------------------------------------------------------------

Graph $D_1^{(0)}$




Adjacency matrix:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 5 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 3 & 0 \\ 2 & 2 & 0 & 4 \end{pmatrix}$$

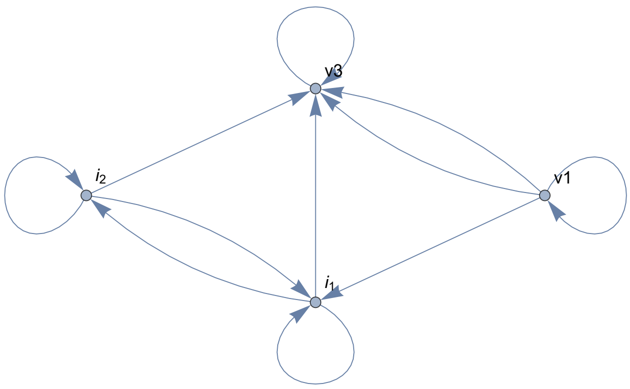
Symmetric adjacency spectrum:

$$\left\{\frac{1}{2}\left(9+\sqrt{41}\right), 4, \frac{1}{2}\left(9-\sqrt{41}\right), 0\right\}$$

$\{7.70156, 4., 1.29844, 0.\}$

------------------------------------------------------------------------

Graph $D_1^{(I)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 6 & 2 & 3 & 3 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 3 & 3 \\ 3 & 1 & 3 & 3 \end{pmatrix}$$

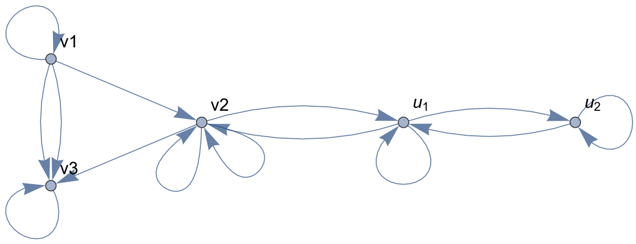
Symmetric adjacency spectrum:

{Root[-6 + 24 #1 - 13 #1^2 + #1^3 &, 3],
Root[-6 + 24 #1 - 13 #1^2 + #1^3 &, 2], Root[-6 + 24 #1 - 13 #1^2 + #1^3 &, 1], 0}

{10.8363, 1.86713, 0.296548, 0.}

---------------------------------------------------------------------------

Graph $D_1^{(C)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 6 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & 2 & 3 \\ 0 & 0 & 2 & 2 & 1 \\ 4 & 1 & 3 & 1 & 6 \end{pmatrix}$$

Symmetric adjacency spectrum:

{Root[-1 + 27 #1 - 103 #1^2 + 83 #1^3 - 18 #1^4 + #1^5 &, 5],
Root[-1 + 27 #1 - 103 #1^2 + 83 #1^3 - 18 #1^4 + #1^5 &, 4],
Root[-1 + 27 #1 - 103 #1^2 + 83 #1^3 - 18 #1^4 + #1^5 &, 3],
Root[-1 + 27 #1 - 103 #1^2 + 83 #1^3 - 18 #1^4 + #1^5 &, 2],
Root[-1 + 27 #1 - 103 #1^2 + 83 #1^3 - 18 #1^4 + #1^5 &, 1]}

{11.5864, 4.6524, 1.42213, 0.294867, 0.0442395}

## Symmetric binary adjacency spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Symmetric binary adjacency spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1"];
adjacencySpecSymmetricBinaryS[d1]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1(S)"];
adjacencySpecSymmetricBinaryS[d1S]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1(R)"];
adjacencySpecSymmetricBinaryS[d1R]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1(O)"];
adjacencySpecSymmetricBinaryS[d1O]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1(I)"];
adjacencySpecSymmetricBinaryS[d1I]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1(C)"];
adjacencySpecSymmetricBinaryS[d1C]
% // N
```

---------------------------------------------------------------------

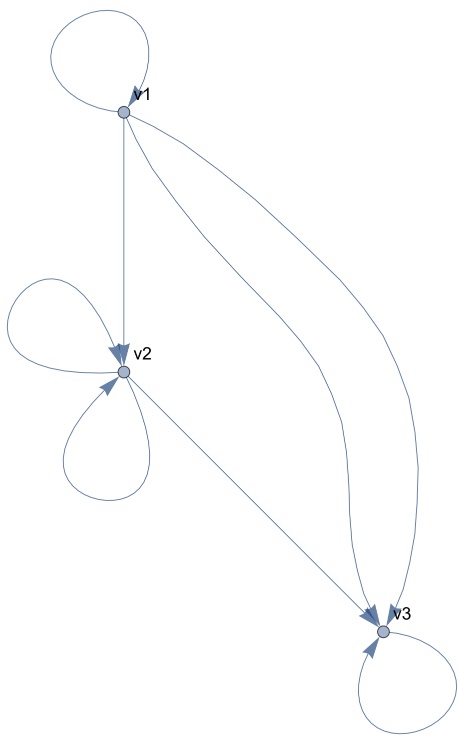
---------------------------------------------------------------------

Symmetric binary adjacency spectra

---------------------------------------------------------------------

---------------------------------------------------------------------

Graph $D_1$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

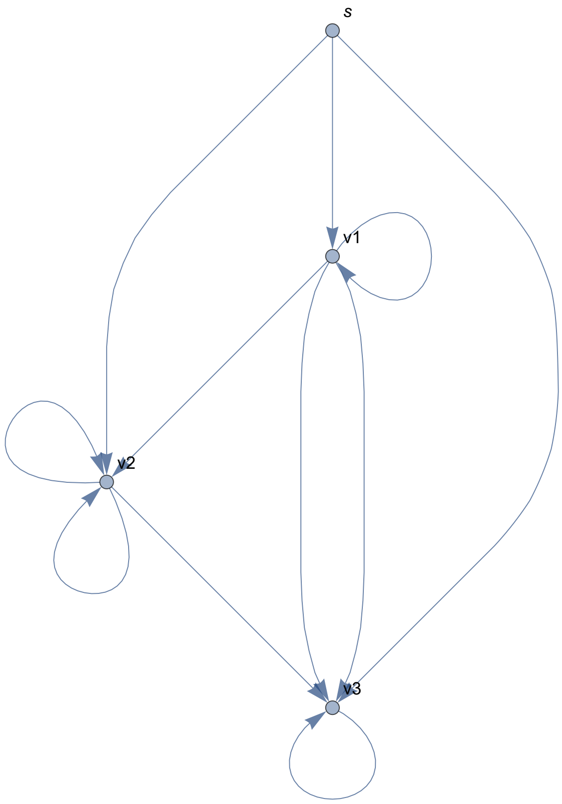
Symmetric binary adjacency spectrum:

{Root[-1 + 5 ♯1 - 6 ♯1^2 + ♯1^3 &, 3],
 Root[-1 + 5 ♯1 - 6 ♯1^2 + ♯1^3 &, 2], Root[-1 + 5 ♯1 - 6 ♯1^2 + ♯1^3 &, 1]}

{5.04892, 0.643104, 0.307979}

------------------------------------------------------------------------

Graph $D_1^{(S)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{pmatrix}$$

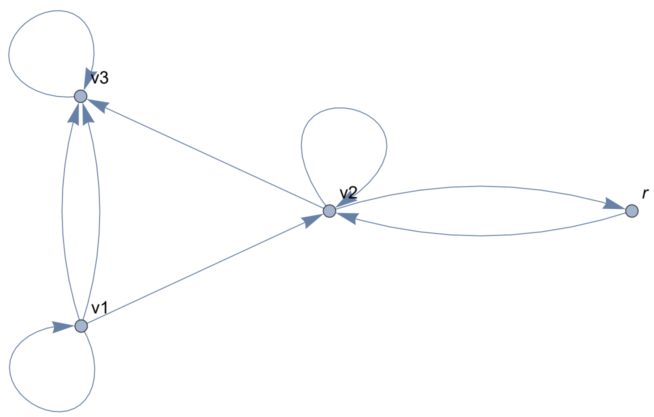
Symmetric binary adjacency spectrum:

{Root[-2 + 9 #1 - 9 #1^2 + #1^3 &, 3],
Root[-2 + 9 #1 - 9 #1^2 + #1^3 &, 2], Root[-2 + 9 #1 - 9 #1^2 + #1^3 &, 1], 0}

{7.89167, 0.785825, 0.322504, 0.}

---------------------------------------------------------------------------

Graph $D_1^{(R)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 3 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

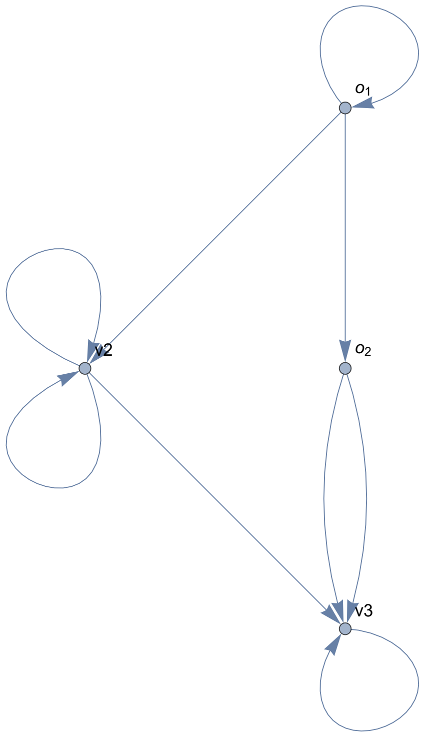
Symmetric binary adjacency spectrum:

$\{3 + 2\sqrt{2}, 1, 1, 3 - 2\sqrt{2}\}$

{5.82843, 1., 1., 0.171573}

------------------------------------------------------------------

Graph $D_1^{(O)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

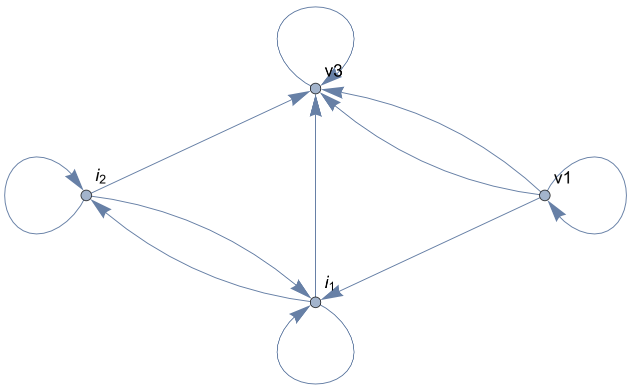
Symmetric binary adjacency spectrum:

$$\left\{4, \frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{5}\right), 0\right\}$$

{4., 2.61803, 0.381966, 0.}

------------------------------------------------------------------------

Graph $D_1^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 3 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 3 \\ 2 & 1 & 3 & 3 \end{pmatrix}$$

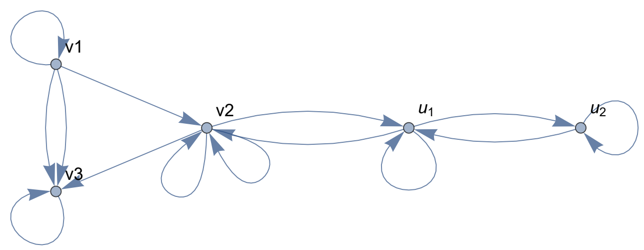
Symmetric binary adjacency spectrum:

{Root[-6 + 16 #1 - 10 #1^2 + #1^3 &, 3],
Root[-6 + 16 #1 - 10 #1^2 + #1^3 &, 2], Root[-6 + 16 #1 - 10 #1^2 + #1^3 &, 1], 0}

{8.12071, 1.31922, 0.560067, 0.}

--------------------------------------------------------------------------------

Graph $D_1^{(C)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 3 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & 2 & 2 \\ 0 & 0 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 3 \end{pmatrix}$$

Symmetric binary adjacency spectrum:

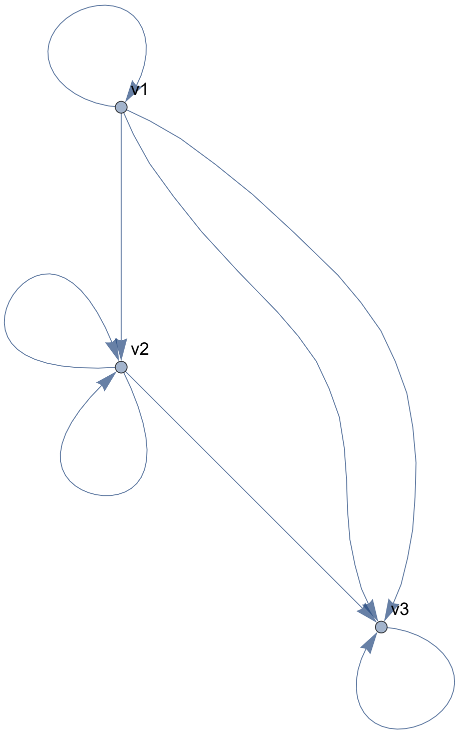
{Root[-1 + 14 #1 - 41 #1^2 + 40 #1^3 - 12 #1^4 + #1^5 &, 5],
Root[-1 + 14 #1 - 41 #1^2 + 40 #1^3 - 12 #1^4 + #1^5 &, 4],
Root[-1 + 14 #1 - 41 #1^2 + 40 #1^3 - 12 #1^4 + #1^5 &, 3],
Root[-1 + 14 #1 - 41 #1^2 + 40 #1^3 - 12 #1^4 + #1^5 &, 2],
Root[-1 + 14 #1 - 41 #1^2 + 40 #1^3 - 12 #1^4 + #1^5 &, 1]}

{7.19584, 3.35194, 0.844535, 0.511755, 0.0959274}

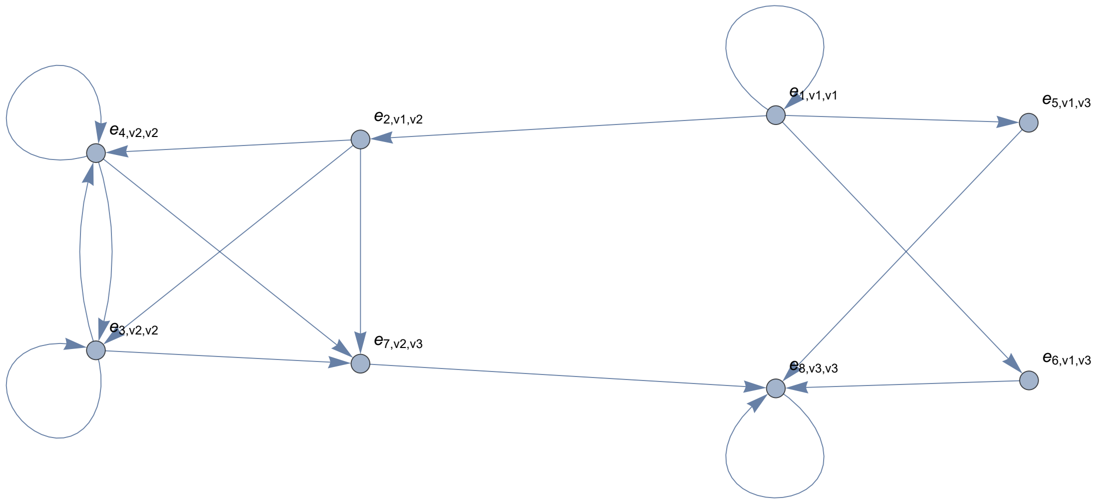
## Line adjacency spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Line adjacency spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_1"];
lineAdjacencySpecS[d1]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_1^(S)"];
lineAdjacencySpecS[d1S]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_1^(I)"];
lineAdjacencySpecS[d1I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_1^(R)"];
lineAdjacencySpecS[d1R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_1^(O)"];
lineAdjacencySpecS[d1O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_1^(C)"];
lineAdjacencySpecS[d1C]
% // N
```

```
------------------------------------------------------------------------------

------------------------------------------------------------------------------

Line adjacency spectra

------------------------------------------------------------------------------

------------------------------------------------------------------------------

Graph D_1
```




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

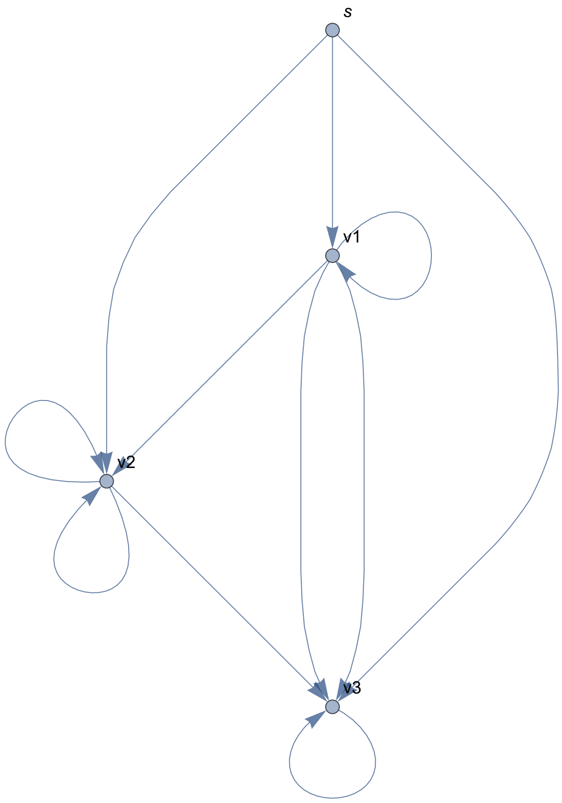
Line adjacency spectrum:

{2, 1, 1, 0, 0, 0, 0, 0}

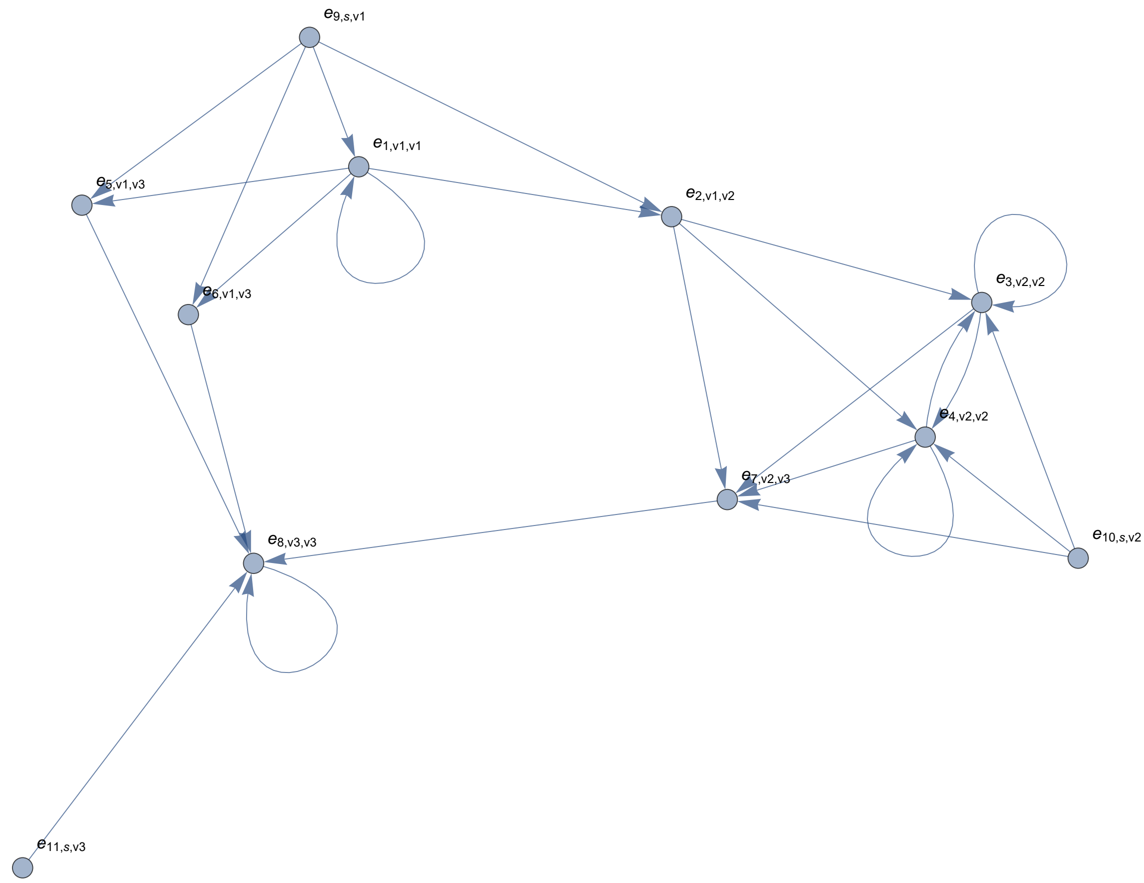
{2., 1., 1., 0., 0., 0., 0., 0.}

----------------------------------------------------------------------

Graph $D_1^{(S)}$




Line graph:




Line adjacency matrix:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

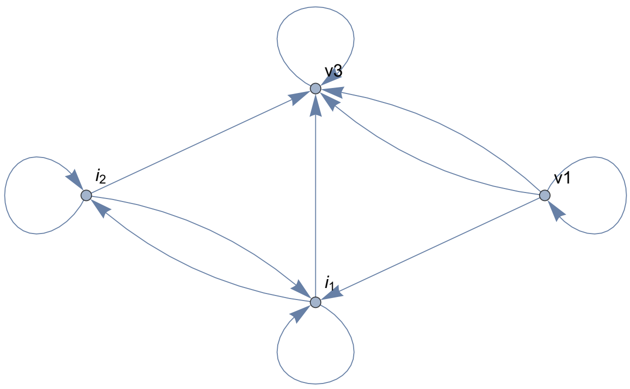
Line adjacency spectrum:

{2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0}

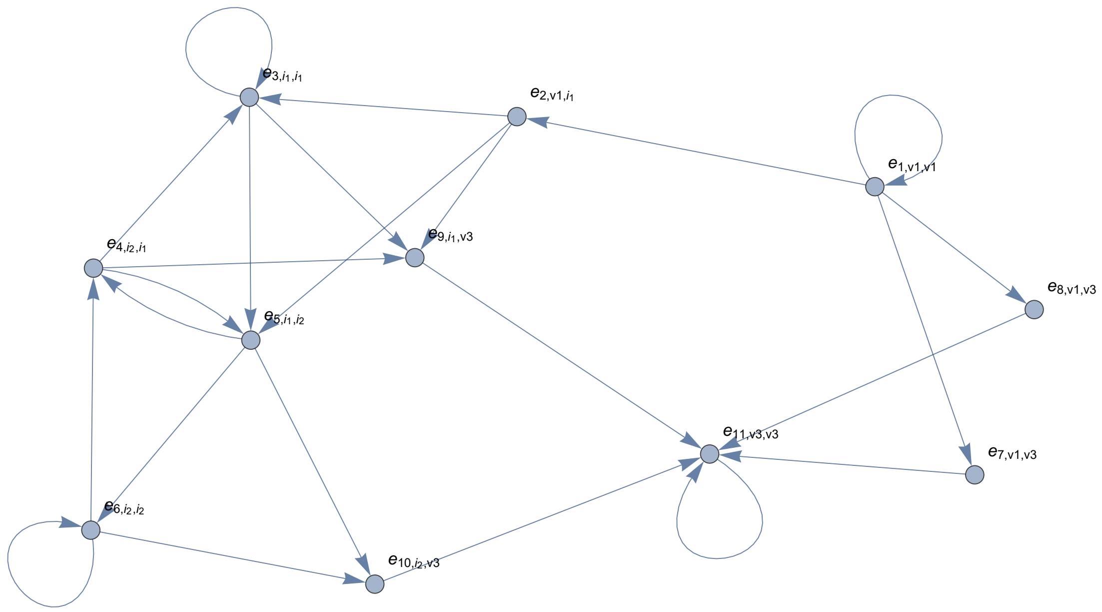
{2., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0.}

---

Graph $D_1^{(I)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

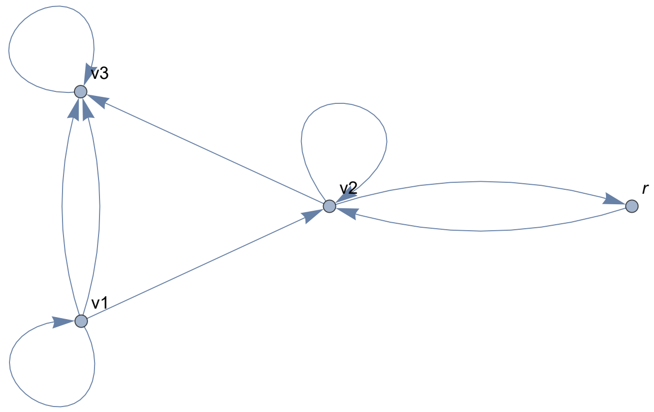
Line adjacency spectrum:

{2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0}

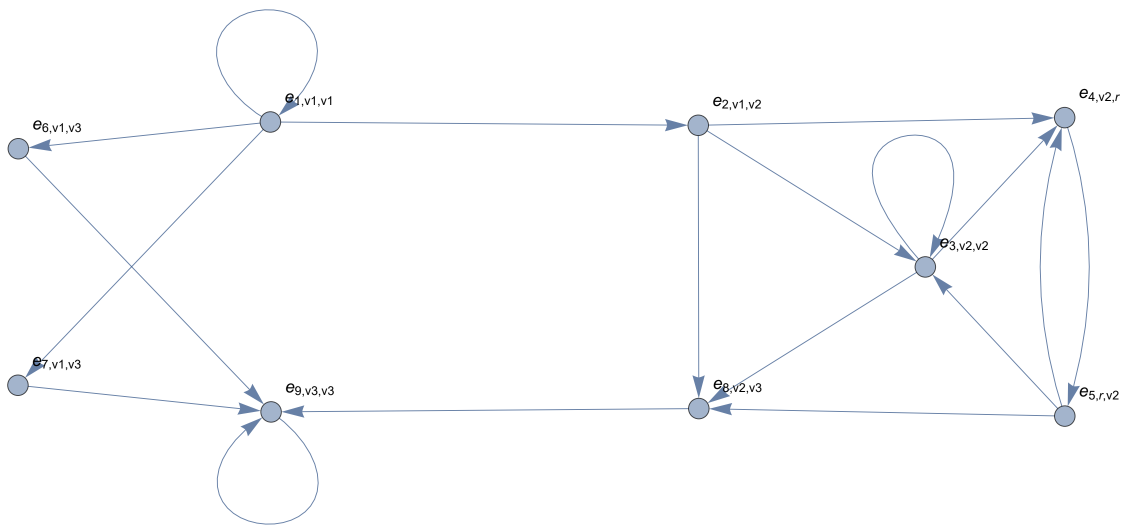
{2., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0.}

---

Graph $D_1^{(R)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

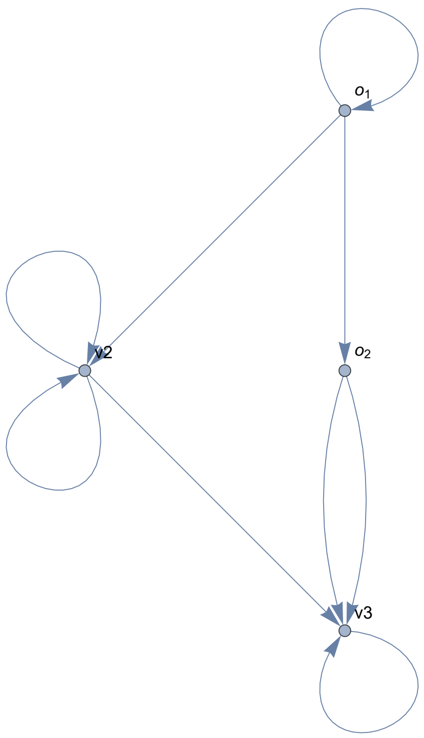
Line adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), 1, 1, \frac{1}{2}\left(1-\sqrt{5}\right), 0, 0, 0, 0, 0\right\}$$

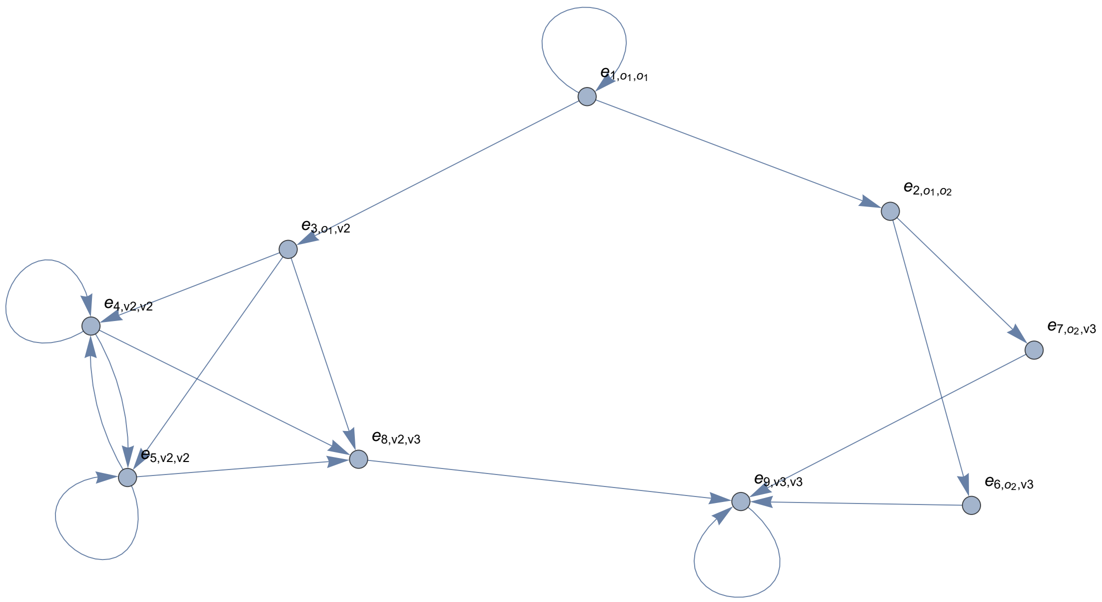
$\{1.61803, 1., 1., -0.618034, 0., 0., 0., 0., 0.\}$

---

Graph $D_1^{(O)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

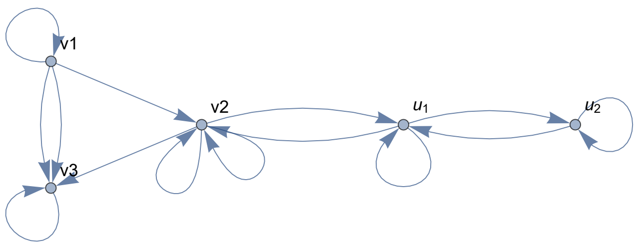
Line adjacency spectrum:

{2, 1, 1, 0, 0, 0, 0, 0, 0}

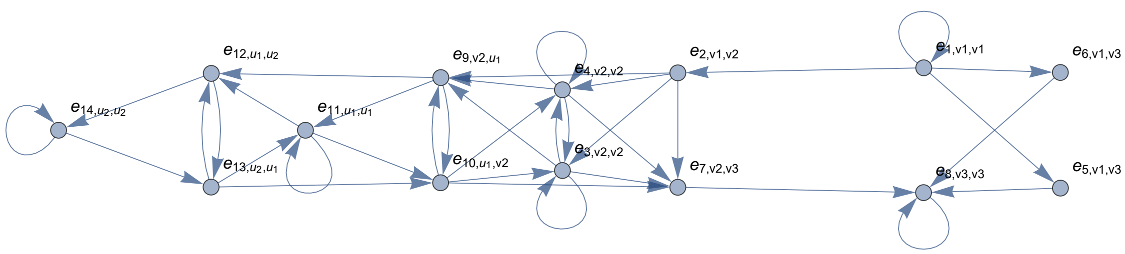
{2., 1., 1., 0., 0., 0., 0., 0., 0.}

---------------------------------------------------------------------------

Graph $D_1^{(C)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Line adjacency spectrum:

```
{Root[1 + 3 #1 - 4 #1^2 + #1^3 &, 3], Root[1 + 3 #1 - 4 #1^2 + #1^3 &, 2],
 1, 1, Root[1 + 3 #1 - 4 #1^2 + #1^3 &, 1], 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

{2.80194, 1.44504, 1., 1., -0.24698, 0., 0., 0., 0., 0., 0., 0., 0., 0.}

## Hermitian adjacency spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Hermitian adjacency spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1"];
hermitianAdjacencySpectrumS[d1]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(S)"];
hermitianAdjacencySpectrumS[d1S]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(R)"];
hermitianAdjacencySpectrumS[d1R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(O)"];
hermitianAdjacencySpectrumS[d1O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(I)"];
hermitianAdjacencySpectrumS[d1I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D1(C)"];
hermitianAdjacencySpectrumS[d1C]
% // N
```

-----------------------------------------------------------------------

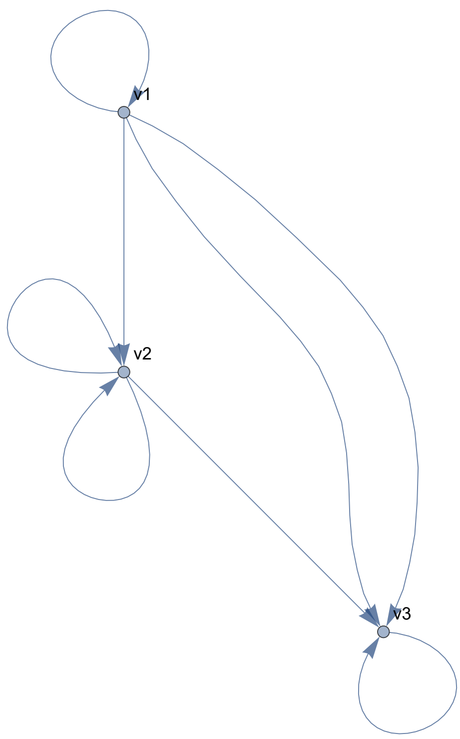
-----------------------------------------------------------------------

Hermitian adjacency spectra

-----------------------------------------------------------------------

-----------------------------------------------------------------------

Graph $D_1$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & i \\ -i & 1 & i \\ -i & -i & 1 \end{pmatrix}$$

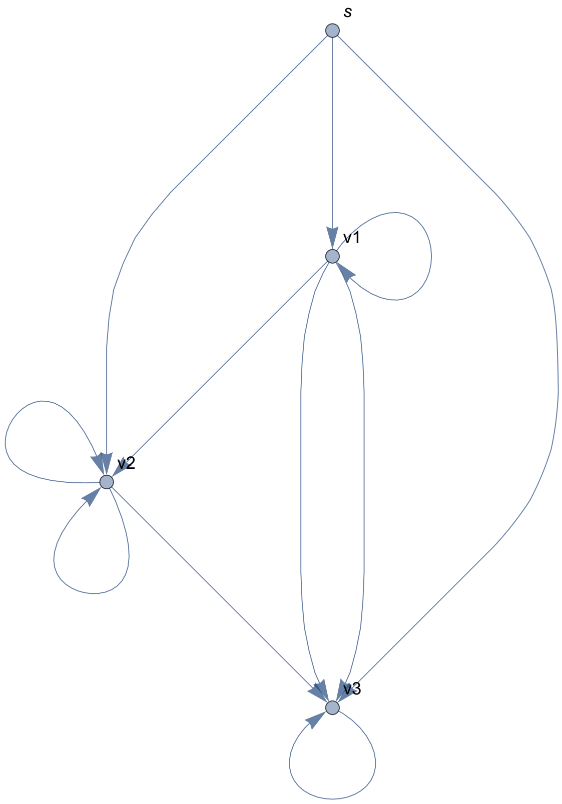
Hermitian adjacency spectrum:

$\{1+\sqrt{3}, 1, 1-\sqrt{3}\}$

{2.73205, 1., -0.732051}

---------------------------------------------------------------------------

Graph $D_1^{(S)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & i & -i \\ -i & 1 & i & -i \\ -i & -i & 1 & -i \\ i & i & i & 0 \end{pmatrix}$$

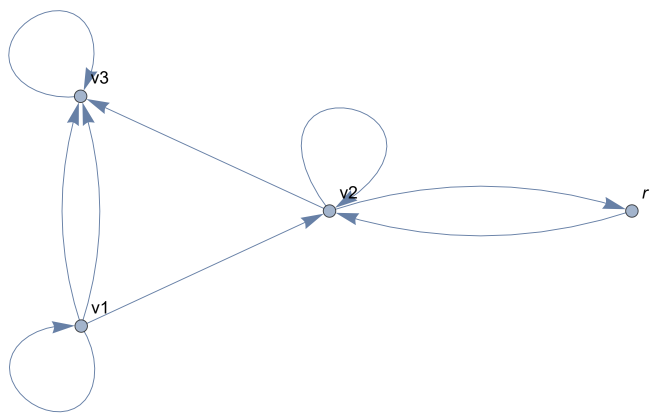
Hermitian adjacency spectrum:

{Root[-2 + 8 #1 - 3 #1^2 - 3 #1^3 + #1^4 &, 4], Root[-2 + 8 #1 - 3 #1^2 - 3 #1^3 + #1^4 &, 1],
Root[-2 + 8 #1 - 3 #1^2 - 3 #1^3 + #1^4 &, 3], Root[-2 + 8 #1 - 3 #1^2 - 3 #1^3 + #1^4 &, 2]}

{3.22001, -1.74108, 1.23136, 0.289713}

---------------------------------------------------------------------------

Graph $D_1^{(R)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & i & 0 \\ -i & 1 & i & 1 \\ -i & -i & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

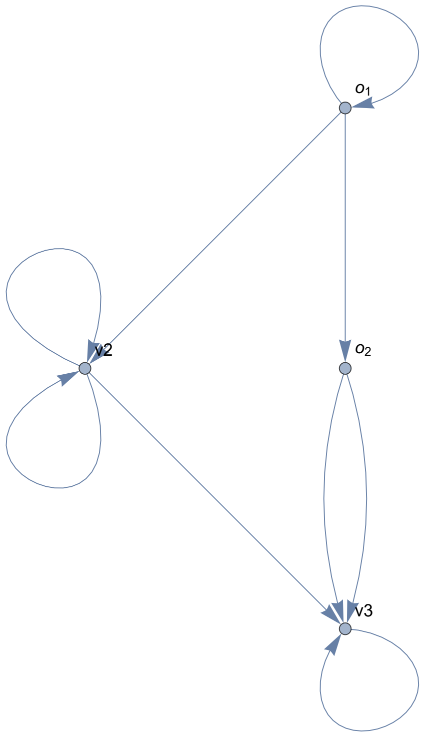
Hermitian adjacency spectrum:

{Root[4 - #1 - 3 #1^2 + #1^3 &, 3],
Root[4 - #1 - 3 #1^2 + #1^3 &, 2], Root[4 - #1 - 3 #1^2 + #1^3 &, 1], 0}

{2.86081, 1.2541, -1.11491, 0.}

---------------------------------------------------------------------------

Graph $D_1^{(O)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & -i & 0 \\ -i & 1 & 0 & -i \\ i & 0 & 1 & i \\ 0 & i & -i & 0 \end{pmatrix}$$

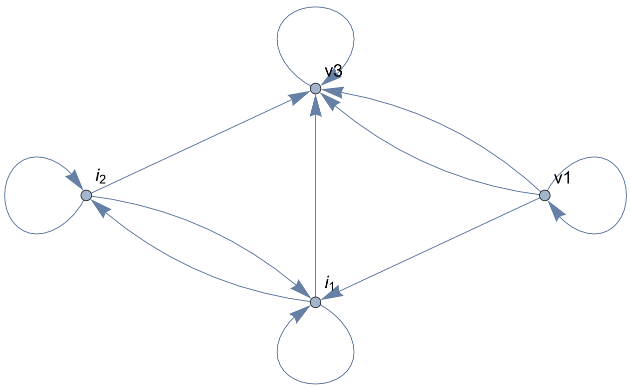
Hermitian adjacency spectrum:

{Root[2 - 3 #1 - 2 #1^2 + #1^3 &, 3],
 Root[2 - 3 #1 - 2 #1^2 + #1^3 &, 1], 1, Root[2 - 3 #1 - 2 #1^2 + #1^3 &, 2]}

{2.81361, -1.34292, 1., 0.529317}

---

Graph $D_1^{(I)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & i & 0 \\ -i & 1 & -i & -i \\ -i & i & 1 & 1 \\ 0 & i & 1 & 1 \end{pmatrix}$$

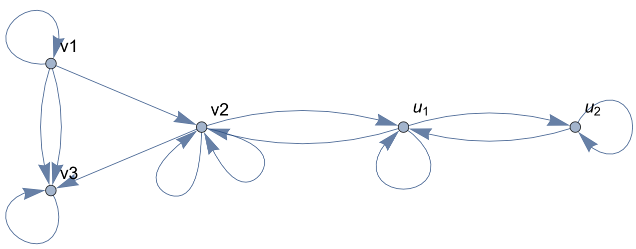
Hermitian adjacency spectrum:

{Root[4 + #1 − 4 #1^2 + #1^3 &, 3],
Root[4 + #1 − 4 #1^2 + #1^3 &, 2], Root[4 + #1 − 4 #1^2 + #1^3 &, 1], 0}

{3.34292, 1.47068, −0.813607, 0.}

---

Graph $D_1^{(C)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & 0 & 0 & i \\ -i & 1 & 0 & 0 & -i \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ -i & i & 1 & 0 & 1 \end{pmatrix}$$

Hermitian adjacency spectrum:

{3, 2, −1, 1, 0}

{3., 2., −1., 1., 0.}

## Skew adjacency spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Skew adjacency spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1"];
skewAdjacencySpecS[d1]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(S)"];
skewAdjacencySpecS[d1S]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(R)"];
skewAdjacencySpecS[d1R]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(O)"];
skewAdjacencySpecS[d1O]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(I)"];
skewAdjacencySpecS[d1I]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(C)"];
skewAdjacencySpecS[d1C]
% // N
```

---------------------------------------------------------------------------

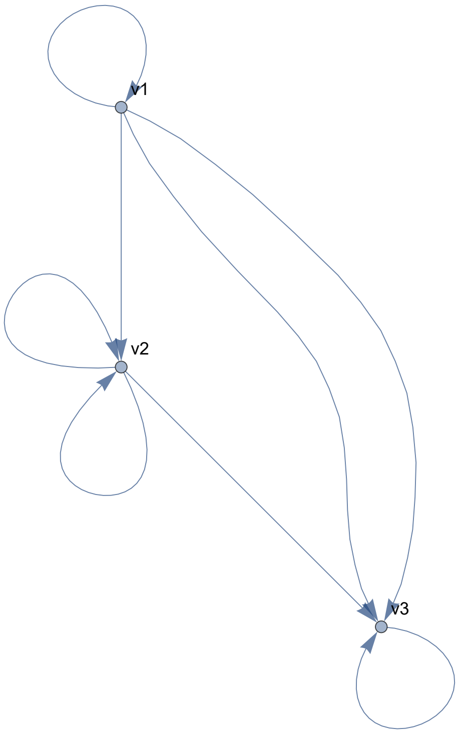
---------------------------------------------------------------------------

Skew adjacency spectra

---------------------------------------------------------------------------

---------------------------------------------------------------------------

Graph $D_1$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

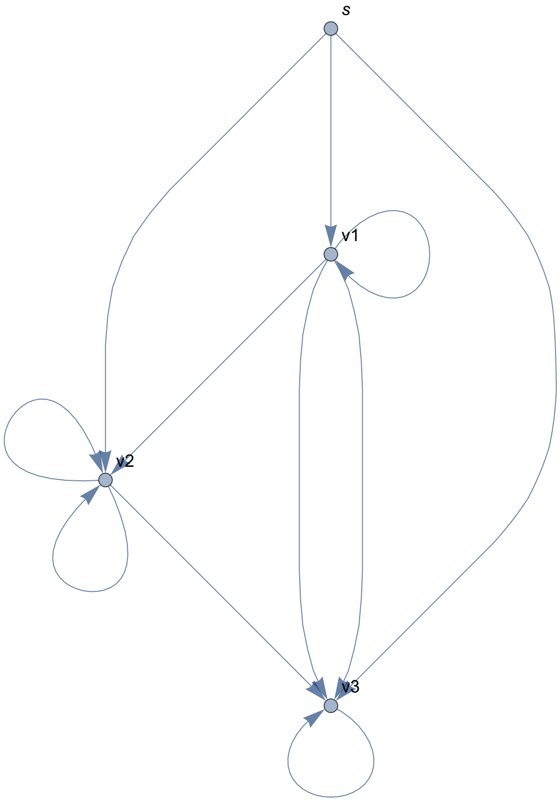
Skew adjacency spectrum:

$\{i\sqrt{6}, -i\sqrt{6}, 0\}$

$\{0. + 2.44949\, i, 0. - 2.44949\, i, 0.\}$

------------------------------------------------------------------------

Graph $D_1^{(S)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 2 & -1 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

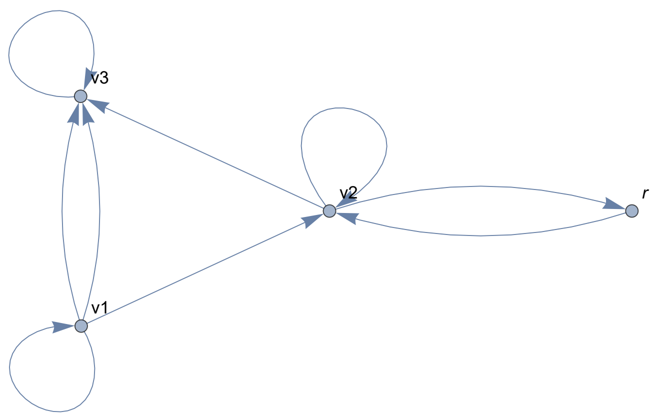
Skew adjacency spectrum:

$\{3\,i, -3\,i, 0, 0\}$

$\{0.+3.\,i, 0.-3.\,i, 0., 0.\}$

--------------------------------------------------------------------------------

Graph $D_1^{(R)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

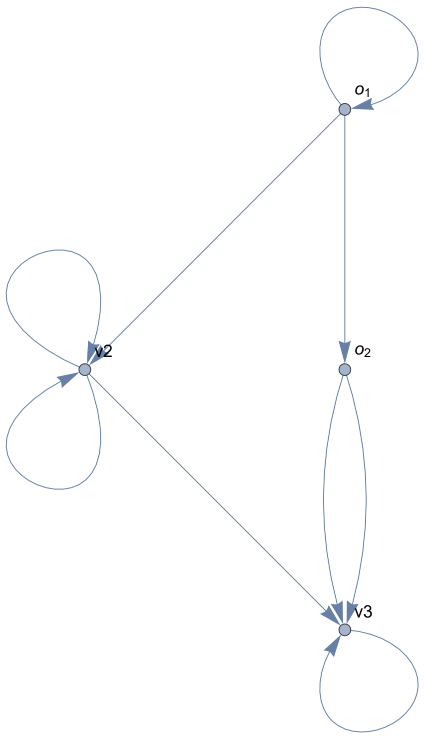
Skew adjacency spectrum:

$\{i\sqrt{6}, -i\sqrt{6}, 0, 0\}$

$\{0. + 2.44949\, i, 0. - 2.44949\, i, 0., 0.\}$

-------------------------------------------------------------------------------

Graph $D_1^{(O)}$




Adjacency matrix:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix}$$

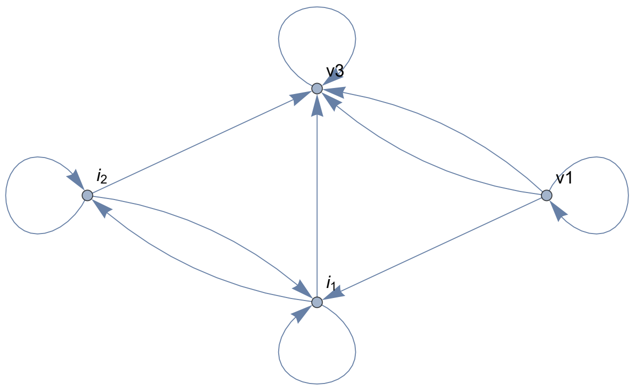
Skew adjacency spectrum:

$$\left\{ i \sqrt{\frac{1}{2}\left(7+3\sqrt{5}\right)},\ -i \sqrt{\frac{1}{2}\left(7+3\sqrt{5}\right)},\ i \sqrt{\frac{7}{2}-\frac{3\sqrt{5}}{2}},\ -i \sqrt{\frac{7}{2}-\frac{3\sqrt{5}}{2}} \right\}$$

$$\{0. + 2.61803\, i,\ 0. - 2.61803\, i,\ 0. + 0.381966\, i,\ 0. - 0.381966\, i\}$$

---------------------------------------------------------------------------

Graph $D_1^{(I)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

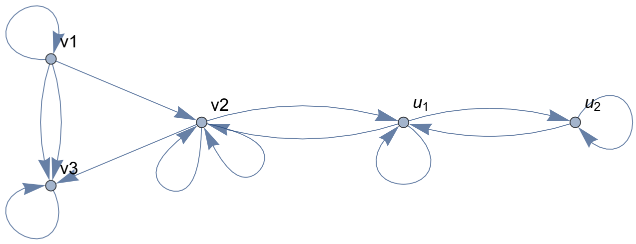
Skew adjacency spectrum:

$$\left\{ i \sqrt{\frac{1}{2}\left(7+3\sqrt{5}\right)},\ -i \sqrt{\frac{1}{2}\left(7+3\sqrt{5}\right)},\ i \sqrt{\frac{7}{2}-\frac{3\sqrt{5}}{2}},\ -i \sqrt{\frac{7}{2}-\frac{3\sqrt{5}}{2}} \right\}$$

$$\{0. + 2.61803\, i,\ 0. - 2.61803\, i,\ 0. + 0.381966\, i,\ 0. - 0.381966\, i\}$$

------------------------------------------------------------------------

Graph $D_1^{(C)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Skew adjacency spectrum:

$\{i\sqrt{6}, -i\sqrt{6}, 0, 0, 0\}$

$\{0. + 2.44949\, i, 0. - 2.44949\, i, 0., 0., 0.\}$

## Binary skew adjacency spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Binary skew adjacency spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1"];
skewAdjacencySpecBinaryS[d1]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(S)"];
skewAdjacencySpecBinaryS[d1S]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(R)"];
skewAdjacencySpecBinaryS[d1R]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(O)"];
skewAdjacencySpecBinaryS[d1O]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(I)"];
skewAdjacencySpecBinaryS[d1I]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(C)"];
skewAdjacencySpecBinaryS[d1C]
% // N
```

---------------------------------------------------------------------------

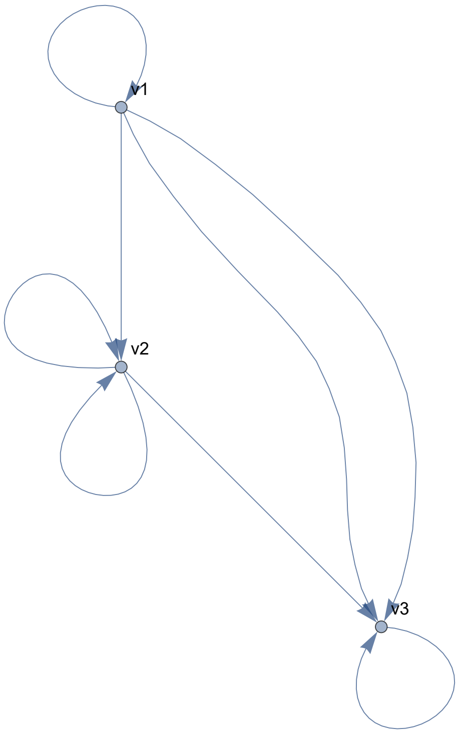
---------------------------------------------------------------------------

Binary skew adjacency spectra

---------------------------------------------------------------------------

---------------------------------------------------------------------------

Graph $D_1$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

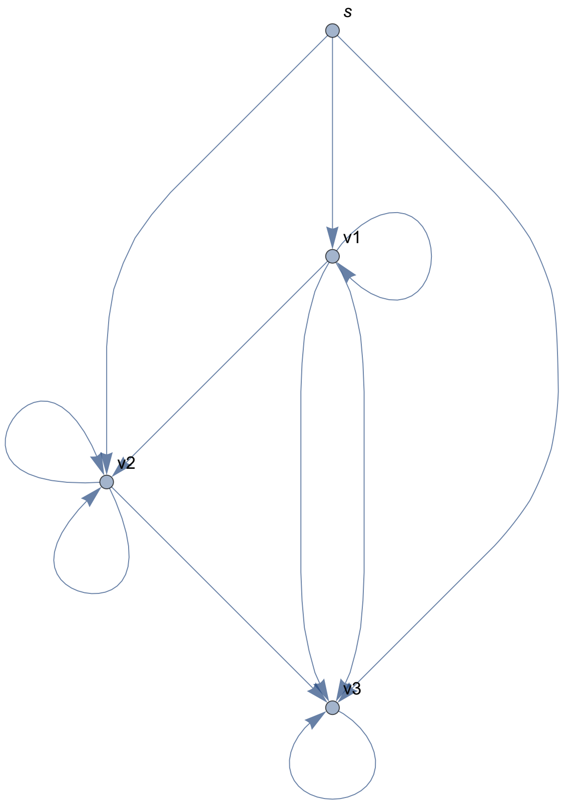
Binary skew adjacency spectrum:

$\{i\sqrt{3}, -i\sqrt{3}, 0\}$

$\{0. + 1.73205\, i, 0. - 1.73205\, i, 0.\}$

------------------------------------------------------------------------

Graph $D_1^{(S)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

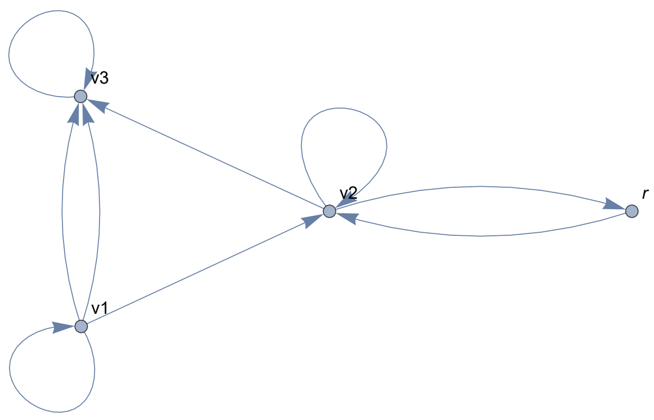
Binary skew adjacency spectrum:

$$\left\{ i\sqrt{3+2\sqrt{2}},\ -i\sqrt{3+2\sqrt{2}},\ i\sqrt{3-2\sqrt{2}},\ -i\sqrt{3-2\sqrt{2}} \right\}$$

$$\{0. + 2.41421\, i,\ 0. - 2.41421\, i,\ 0. + 0.414214\, i,\ 0. - 0.414214\, i\}$$

---

Graph $D_1^{(R)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

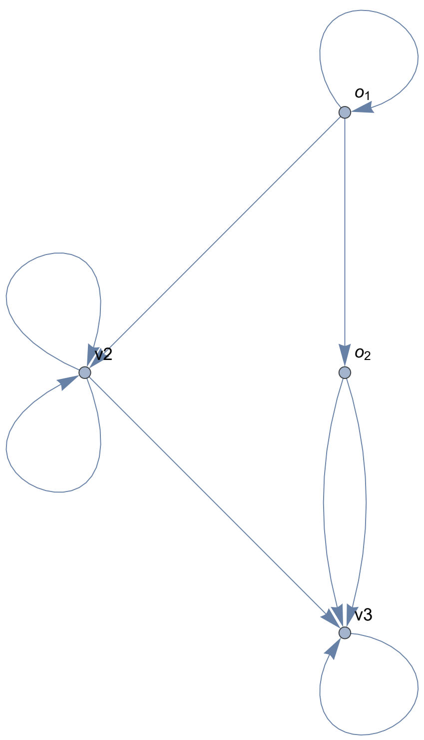
Binary skew adjacency spectrum:

$\{i\sqrt{3}, -i\sqrt{3}, 0, 0\}$

$\{0. + 1.73205\, i, 0. - 1.73205\, i, 0., 0.\}$

----------------------------------------------------------------------

Graph $D_1^{(O)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

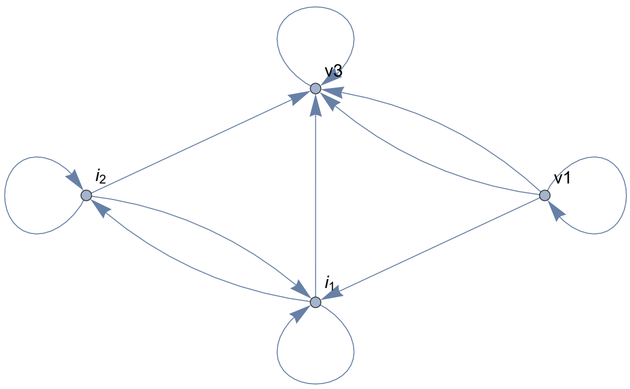
Binary skew adjacency spectrum:

$\{2\,i, -2\,i, 0, 0\}$

$\{0. + 2.\,i, 0. - 2.\,i, 0., 0.\}$

-----------------------------------------------------------------------------

Graph $D_1^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

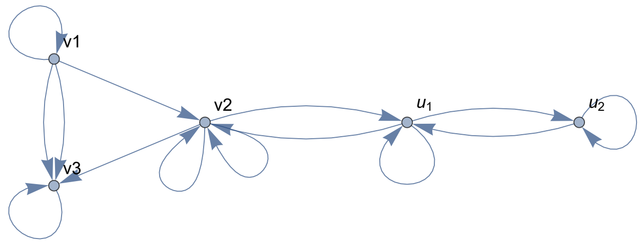
Binary skew adjacency spectrum:

$$\left\{ i\sqrt{2+\sqrt{3}},\ -i\sqrt{2+\sqrt{3}},\ i\sqrt{2-\sqrt{3}},\ -i\sqrt{2-\sqrt{3}} \right\}$$

$$\{0. + 1.93185\, i,\ 0. - 1.93185\, i,\ 0. + 0.517638\, i,\ 0. - 0.517638\, i\}$$

------------------------------------------------------------------------

Graph $D_1^{(C)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency spectrum:

$\{i\sqrt{3}, -i\sqrt{3}, 0, 0, 0\}$

$\{0. + 1.73205\, i, 0. - 1.73205\, i, 0., 0., 0.\}$

## Skew Laplace spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Skew Laplace spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1"];
skewLaplaceSpecS[d1]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(S)"];
skewLaplaceSpecS[d1S]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(I)"];
skewLaplaceSpecS[d1I]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(R)"];
skewLaplaceSpecS[d1R]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(O)"];
skewLaplaceSpecS[d1O]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_1^(C)"];
skewLaplaceSpecS[d1C]
% // N
```

----------------------------------------------------------------------

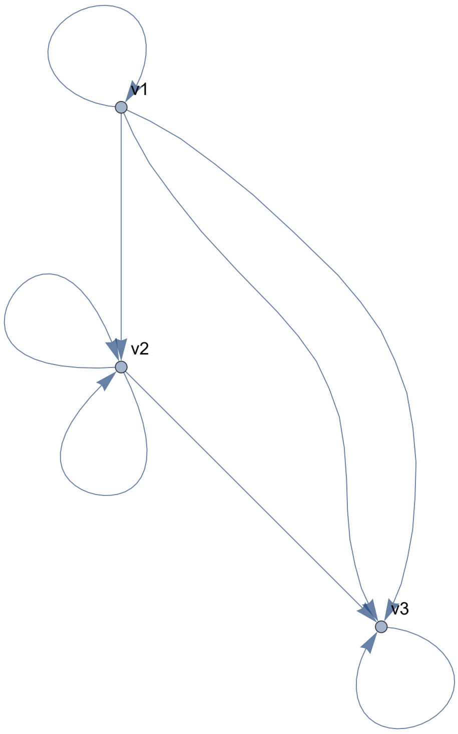
----------------------------------------------------------------------

Skew Laplace spectra

----------------------------------------------------------------------

----------------------------------------------------------------------

Graph $D_1$




Skew Laplacian matrix

$$\begin{pmatrix} 3 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & -3 \end{pmatrix}$$

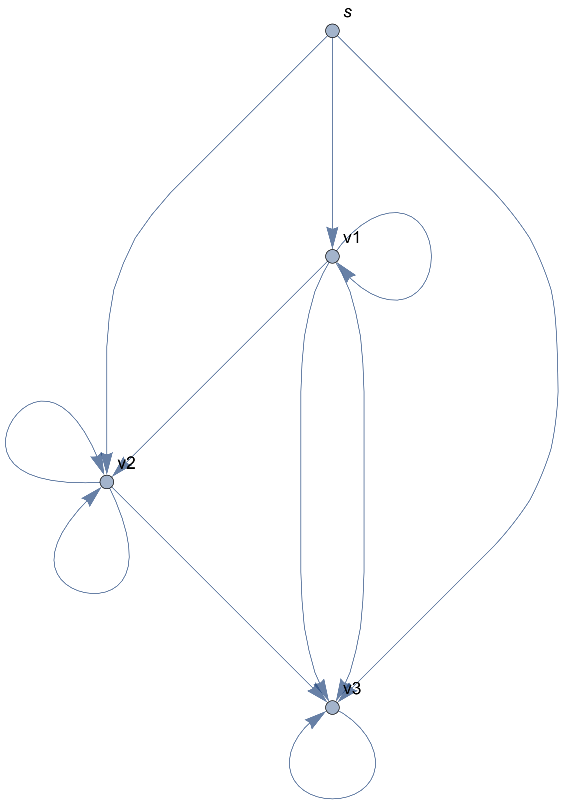
Skew Laplace spectrum

$\{-\sqrt{3}, \sqrt{3}, 0\}$

$\{-1.73205, 1.73205, 0.\}$

-----------------------------------------------------------------------

Graph $D_1^{(S)}$




Skew Laplacian matrix

$$\begin{pmatrix} 2 & -1 & -2 & 1 \\ 1 & -1 & -1 & 1 \\ 2 & 1 & -4 & 1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

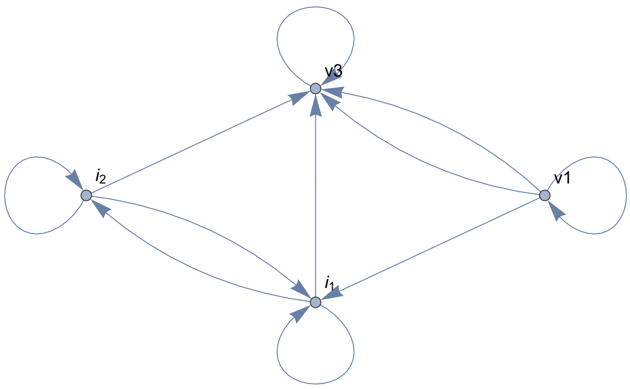
Skew Laplace spectrum

$\{-1-\sqrt{3}, 2, -1+\sqrt{3}, 0\}$

{-2.73205, 2., 0.732051, 0.}

---

Graph $D_1^{(I)}$




Skew Laplacian matrix

$$\begin{pmatrix} 3 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

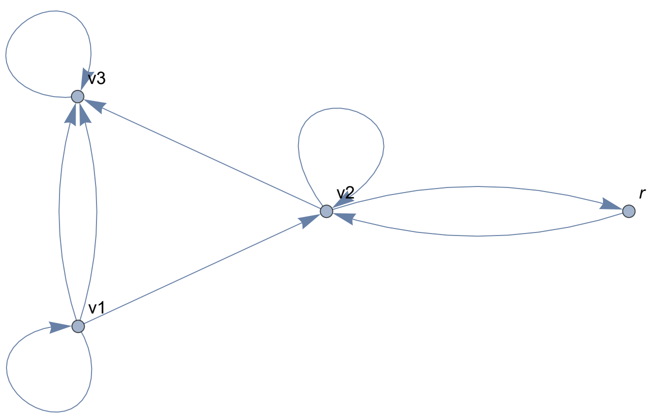
Skew Laplace spectrum

$\left\{-1-\sqrt{3}, 2, -1+\sqrt{3}, 0\right\}$

$\{-2.73205, 2., 0.732051, 0.\}$

---

Graph $D_1^{(R)}$




Skew Laplacian matrix

$$\begin{pmatrix} 3 & -1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

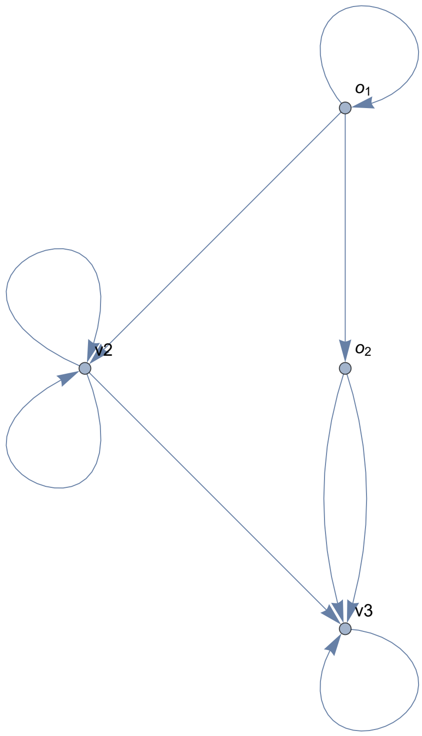
Skew Laplace spectrum

$\left\{-\sqrt{3}, \sqrt{3}, 0, 0\right\}$

$\{-1.73205, 1.73205, 0., 0.\}$

-----------------------------------------------------------------------

Graph $D_1^{(0)}$




Skew Laplacian matrix

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -3 & 0 & 2 \\ -1 & 0 & 2 & -1 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

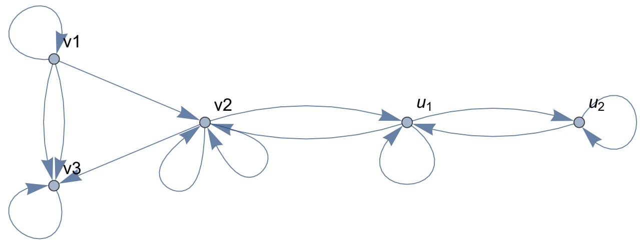
Skew Laplace spectrum

{0, 0, 0, 0}

{0., 0., 0., 0.}

------------------------------------------------------------------------

Graph $D_1^{(C)}$




Skew Laplacian matrix

$$\begin{pmatrix} 3 & -2 & 0 & 0 & -1 \\ 2 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Skew Laplace spectrum

$\{-\sqrt{3}, \sqrt{3}, 0, 0, 0\}$

{-1.73205, 1.73205, 0., 0., 0.}

## Binary skew Laplace spectra

```
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Binary skew Laplace spectra"];
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1"];
skewLaplaceSpecBinaryS[d1]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1(S)"];
skewLaplaceSpecBinaryS[d1S]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1(R)"];
skewLaplaceSpecBinaryS[d1R]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1(O)"];
skewLaplaceSpecBinaryS[d1O]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1(I)"];
skewLaplaceSpecBinaryS[d1I]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1(C)"];
skewLaplaceSpecBinaryS[d1C]
% // N
```

------------------------------------------------------------------------

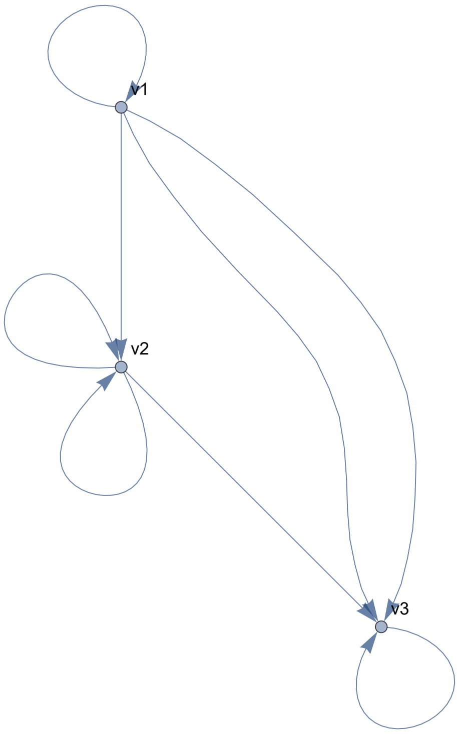
------------------------------------------------------------------------

Binary skew Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_1$




Binary skew Laplacian matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix}$$

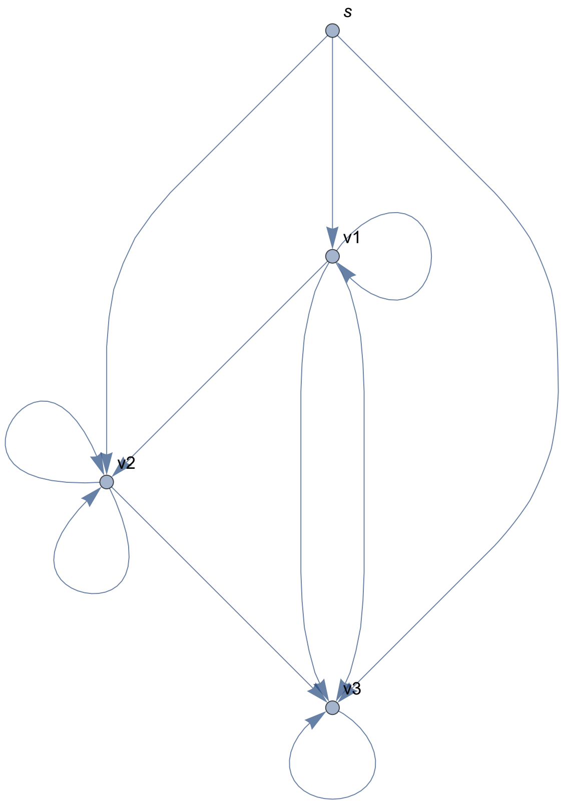
Binary skew Laplace spectrum

{-1, 1, 0}

{-1., 1., 0.}

-----------------------------------------------------------------------

Graph $D_1^{(S)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -3 & 1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

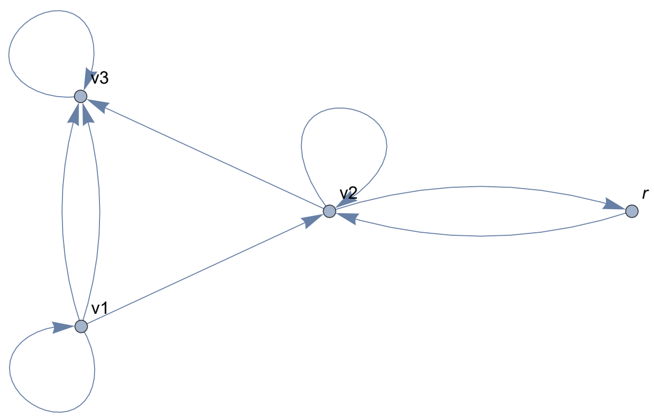
Binary skew Laplace spectrum

{-2, 2, 0, 0}

{-2., 2., 0., 0.}

------------------------------------------------------------------------

Graph $D_1^{(R)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

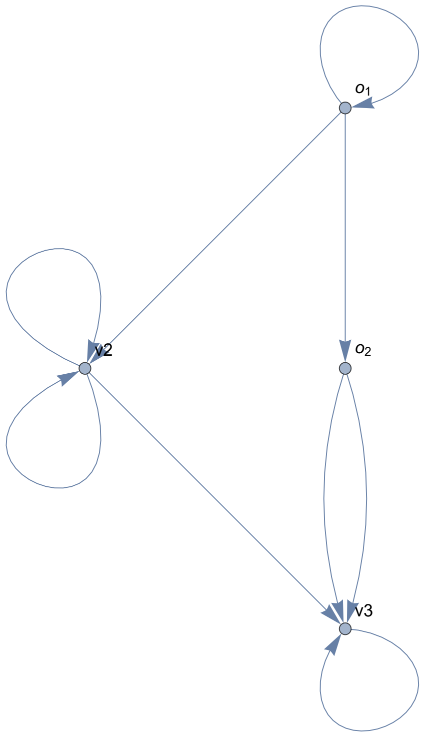
Binary skew Laplace spectrum

{-1, 1, 0, 0}

{-1., 1., 0., 0.}

-----------------------------------------------------------------------

Graph $D_1^{(0)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

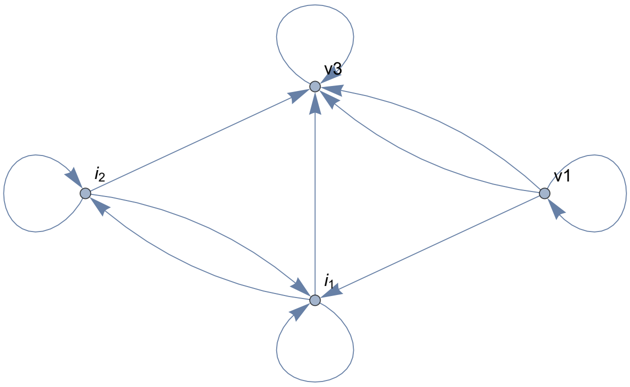
Binary skew Laplace spectrum

{0, 0, 0, 0}

{0., 0., 0., 0.}

---

Graph $D_1^{(I)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

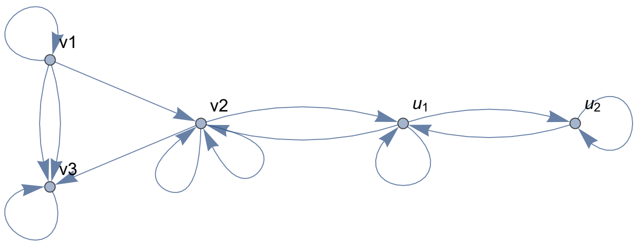
Binary skew Laplace spectrum

{-2, 1, 1, 0}

{-2., 1., 1., 0.}

---

Graph $D_1^{(C)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Binary skew Laplace spectrum

{-1, 1, 0, 0, 0}

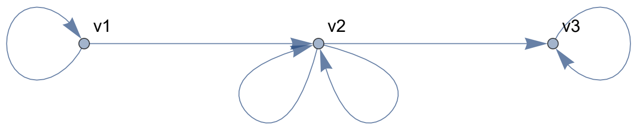
{-1., 1., 0., 0., 0.}

# Example 5.2 (of arXiv:2010.10769 [math.CO] v1)

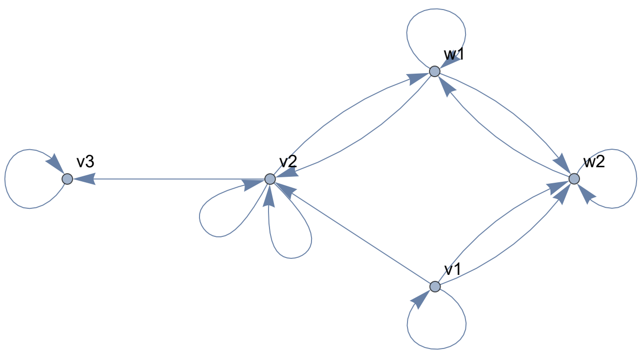
## Definitions of the graphs

```
(* Define the graph D2 *)
Print["Graph D2"];
d2 = Graph[{v1, v2, v3},
  {v1 ⥂ v1, v1 ⥂ v2, v2 ⥂ v2, v2 ⥂ v2, v2 ⥂ v3, v3 ⥂ v3}, VertexLabels → "Name"]

(* Define the graph D1^(P) *)
Print["Graph D1^(P)"];
d2P = Graph[{v1, v2, v3, w1, w2},
  {v1 ⥂ v1, v1 ⥂ v2, v2 ⥂ v2, v2 ⥂ v2, v2 ⥂ v3, v3 ⥂ v3, v2 ⥂ w1, w1 ⥂ v2,
   w1 ⥂ w1, w1 ⥂ w2, w2 ⥂ w1, w2 ⥂ w2, v1 ⥂ w2, v1 ⥂ w2}, VertexLabels → "Name"]
```

Graph $D_2$




Graph $D_1^{(P)}$

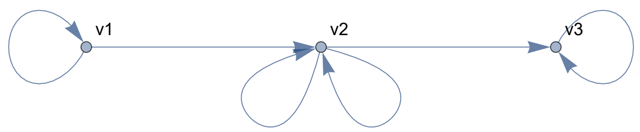


## Laplace spectra

```
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Laplace spectra"];
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Graph D2"];
laplaceSpecS[d2]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D1(P)"];
laplaceSpecS[d2P]
% // N
```

```
------------------------------------------------------------------------

------------------------------------------------------------------------

Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------
```

Graph $D_2$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

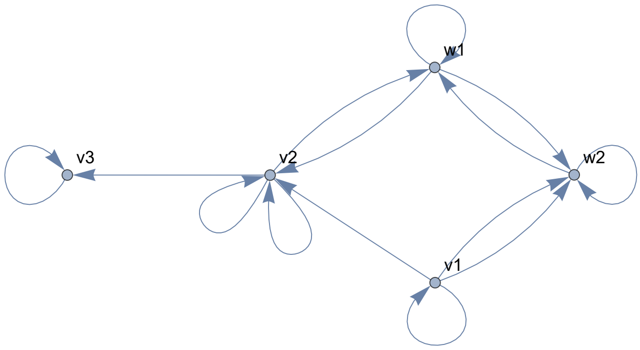
Laplace spectrum:

{3, 1, 0}

{3., 1., 0.}

---------------------------------------------------------------------------

Graph $D_1^{(P)}$




Incidence matrix:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | -1 | 0 | 0 | 1 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 1 | -1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | -1 | -1 |

Laplace spectrum:

{Root[100 - 172 #1 + 85 #1^2 - 16 #1^3 + #1^4 &, 4], Root[100 - 172 #1 + 85 #1^2 - 16 #1^3 + #1^4 &, 3],
Root[100 - 172 #1 + 85 #1^2 - 16 #1^3 + #1^4 &, 2], Root[100 - 172 #1 + 85 #1^2 - 16 #1^3 + #1^4 &, 1], 0}

{7.43874, 4.451, 3.15208, 0.958176, 0.}

# Adjacency spectra

```
Print[
  "---------------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------------"];
Print["Adjacency spectra"];
Print[
  "---------------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------------"];
Print["Graph D2"];
adjacencySpecS[d2]
% // N
Print[
  "---------------------------------------------------------------------------"];
Print["Graph D1(P)"];
adjacencySpecS[d2P]
% // N
```

----------------------------------------------------------------------

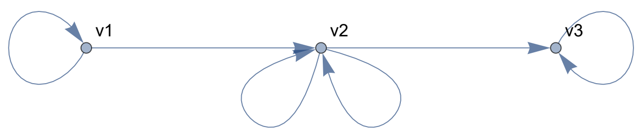
----------------------------------------------------------------------

Adjacency spectra

----------------------------------------------------------------------

----------------------------------------------------------------------

Graph $D_2$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

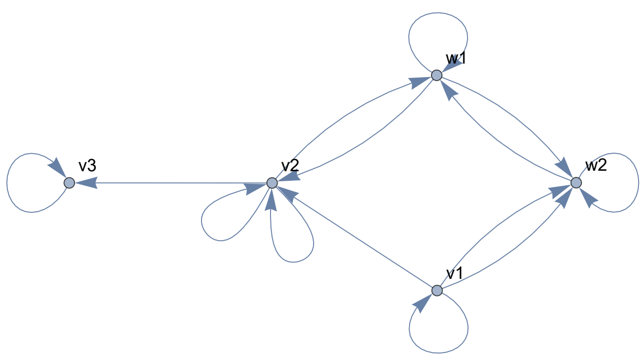
Adjacency spectrum:

{2, 1, 1}

{2., 1., 1.}

----------------------------------------------------------------------

Graph $D_1^{(P)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Adjacency spectrum:

$\{\text{Root}[1 + 3 \#1 - 4 \#1^2 + \#1^3 \&, 3],$
$\text{Root}[1 + 3 \#1 - 4 \#1^2 + \#1^3 \&, 2], 1, 1, \text{Root}[1 + 3 \#1 - 4 \#1^2 + \#1^3 \&, 1]\}$

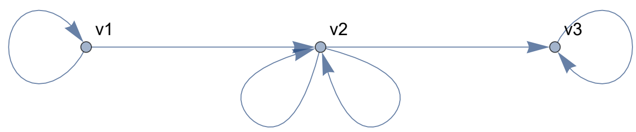
{2.80194, 1.44504, 1., 1., -0.24698}

## Binary adjacency spectra

```
Print[
  "---------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------"];
Print["Binary adjacency spectra"];
Print[
  "---------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------"];
Print["Graph D2"];
adjacencySpecBinaryS[d2]
% // N
Print[
  "---------------------------------------------------------------------"];
Print["Graph D1(P)"];
adjacencySpecBinaryS[d2P]
% // N
```

```
---------------------------------------------------------------------

---------------------------------------------------------------------

Binary adjacency spectra

---------------------------------------------------------------------

---------------------------------------------------------------------
```

Graph $D_2$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

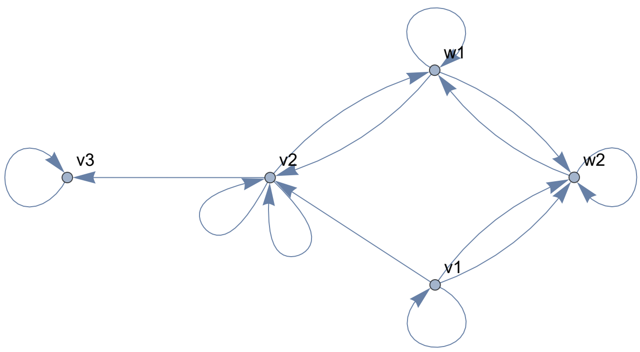
Binary adjacency spectrum:

{1, 1, 1}

{1., 1., 1.}

-----------------------------------------------------------------------

Graph $D_1^{(P)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Binary adjacency spectrum:

$\{1+\sqrt{2}, 1, 1, 1, 1-\sqrt{2}\}$

{2.41421, 1., 1., 1., -0.414214}

## Symmetric adjacency spectra

```
Print[
  "-----------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------"];
Print["Symmetric adjacency spectra"];
Print[
  "-----------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------"];
Print["Graph D₂"];
adjacencySpecSymmetricS[d2]
% // N
Print[
  "-----------------------------------------------------------------------"];
Print["Graph D₁⁽ᴾ⁾"];
adjacencySpecSymmetricS[d2P]
% // N
```

-------------------------------------------------------------------

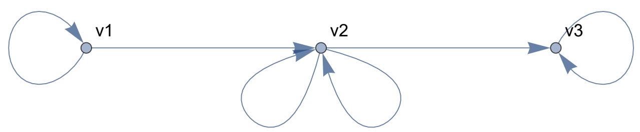
-------------------------------------------------------------------

Symmetric adjacency spectra

-------------------------------------------------------------------

-------------------------------------------------------------------

Graph $D_2$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

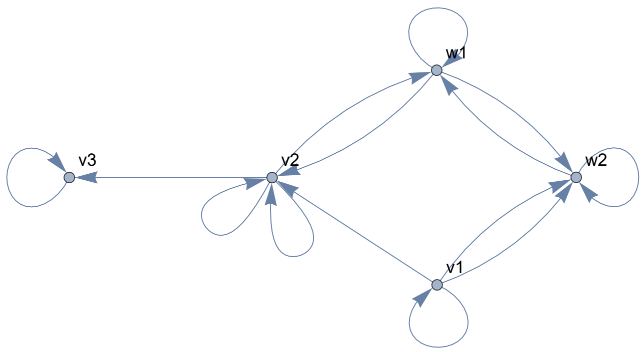
Symmetric adjacency spectrum:

{Root[-4 + 12 ⌗1 - 8 ⌗1^2 + ⌗1^3 &, 3],
Root[-4 + 12 ⌗1 - 8 ⌗1^2 + ⌗1^3 &, 2], Root[-4 + 12 ⌗1 - 8 ⌗1^2 + ⌗1^3 &, 1]}

{6.15633, 1.3691, 0.474572}

---------------------------------------------------------------------

Graph $D_1^{(P)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 6 & 2 & 0 & 3 & 2 \\ 2 & 6 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & 0 & 3 & 2 \\ 2 & 1 & 0 & 2 & 2 \end{pmatrix}$$

Symmetric adjacency spectrum:

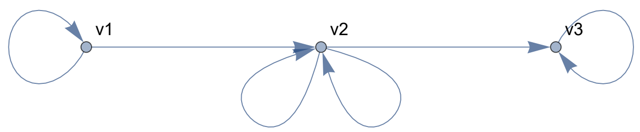
$$\{\text{Root}[-1 + 63\,\#1 - 135\,\#1^2 + 87\,\#1^3 - 18\,\#1^4 + \#1^5\,\&,\,5],$$
$$\text{Root}[-1 + 63\,\#1 - 135\,\#1^2 + 87\,\#1^3 - 18\,\#1^4 + \#1^5\,\&,\,4],$$
$$\text{Root}[-1 + 63\,\#1 - 135\,\#1^2 + 87\,\#1^3 - 18\,\#1^4 + \#1^5\,\&,\,3],$$
$$\text{Root}[-1 + 63\,\#1 - 135\,\#1^2 + 87\,\#1^3 - 18\,\#1^4 + \#1^5\,\&,\,2],$$
$$\text{Root}[-1 + 63\,\#1 - 135\,\#1^2 + 87\,\#1^3 - 18\,\#1^4 + \#1^5\,\&,\,1]\}$$

{11.3293, 4.32428, 1.50561, 0.824316, 0.0164465}

## Symmetric binary adjacency spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Symmetric binary adjacency spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D2"];
adjacencySpecSymmetricBinaryS[d2]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D1(P)"];
adjacencySpecSymmetricBinaryS[d2P]
% // N
```

```
--------------------------------------------------------------------------

--------------------------------------------------------------------------

Symmetric binary adjacency spectra

--------------------------------------------------------------------------

--------------------------------------------------------------------------

Graph D2
```




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

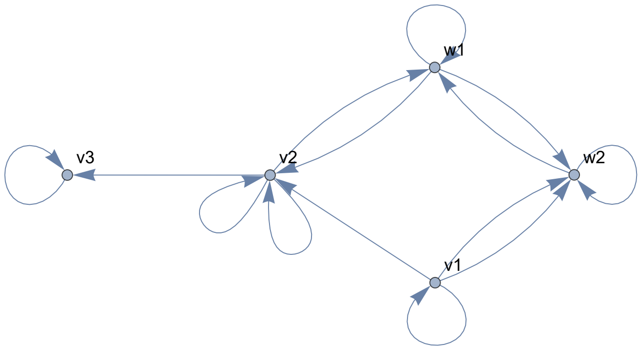
$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Symmetric binary adjacency spectrum:

```
{Root[-1 + 6 #1 - 5 #1^2 + #1^3 &, 3],
 Root[-1 + 6 #1 - 5 #1^2 + #1^3 &, 2], Root[-1 + 6 #1 - 5 #1^2 + #1^3 &, 1]}

{3.24698, 1.55496, 0.198062}
```

---------------------------------------------------------------------------

Graph $D_1^{(P)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 3 & 1 & 0 & 2 & 1 \\ 1 & 3 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 2 & 2 \end{pmatrix}$$

Symmetric binary adjacency spectrum:

$$\{\text{Root}[-1 + 18 \#1 - 48 \#1^2 + 40 \#1^3 - 12 \#1^4 + \#1^5 \&, 5], \\ \text{Root}[-1 + 18 \#1 - 48 \#1^2 + 40 \#1^3 - 12 \#1^4 + \#1^5 \&, 4], \\ \text{Root}[-1 + 18 \#1 - 48 \#1^2 + 40 \#1^3 - 12 \#1^4 + \#1^5 \&, 3], \\ \text{Root}[-1 + 18 \#1 - 48 \#1^2 + 40 \#1^3 - 12 \#1^4 + \#1^5 \&, 2], \\ \text{Root}[-1 + 18 \#1 - 48 \#1^2 + 40 \#1^3 - 12 \#1^4 + \#1^5 \&, 1]\}$$

{7.45504, 2.40912, 1.52115, 0.547884, 0.0668084}

## Line adjacency spectra

```
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Line adjacency spectra"];
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Graph D2"];
lineAdjacencySpecS[d2]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D1(P)"];
lineAdjacencySpecS[d2P]
% // N
```

----------------------------------------------------------------------

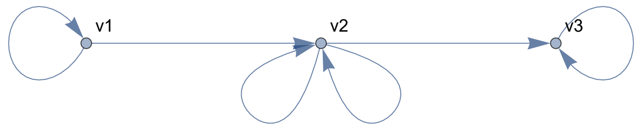
----------------------------------------------------------------------

Line adjacency spectra

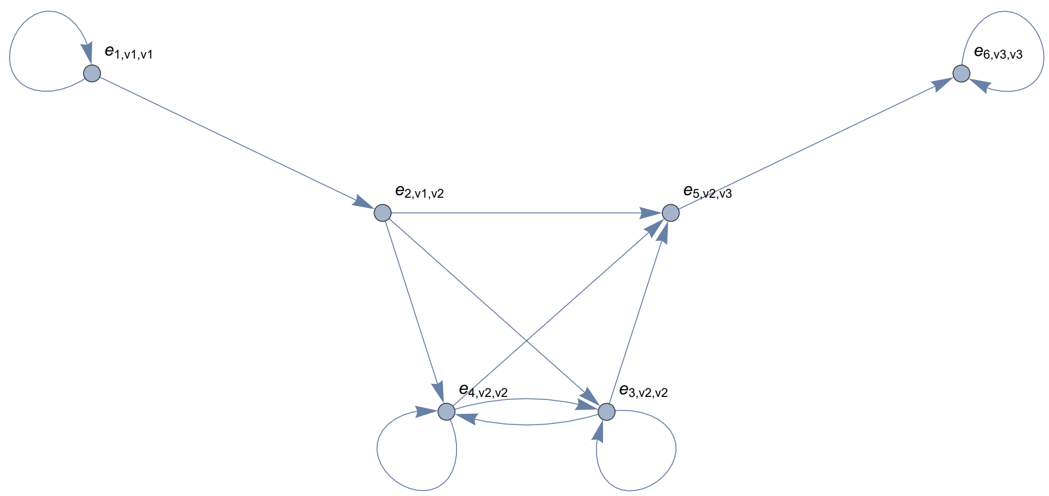
----------------------------------------------------------------------

----------------------------------------------------------------------

Graph $D_2$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

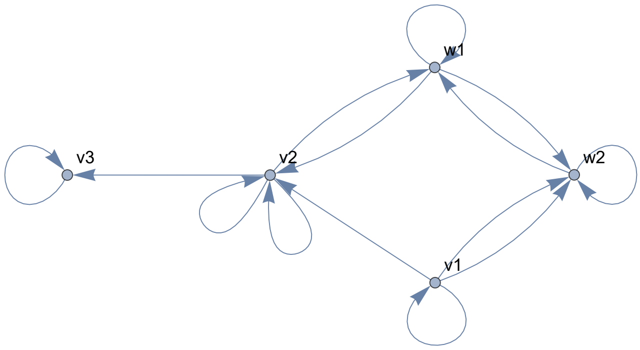
Line adjacency spectrum:

{2, 1, 1, 0, 0, 0}

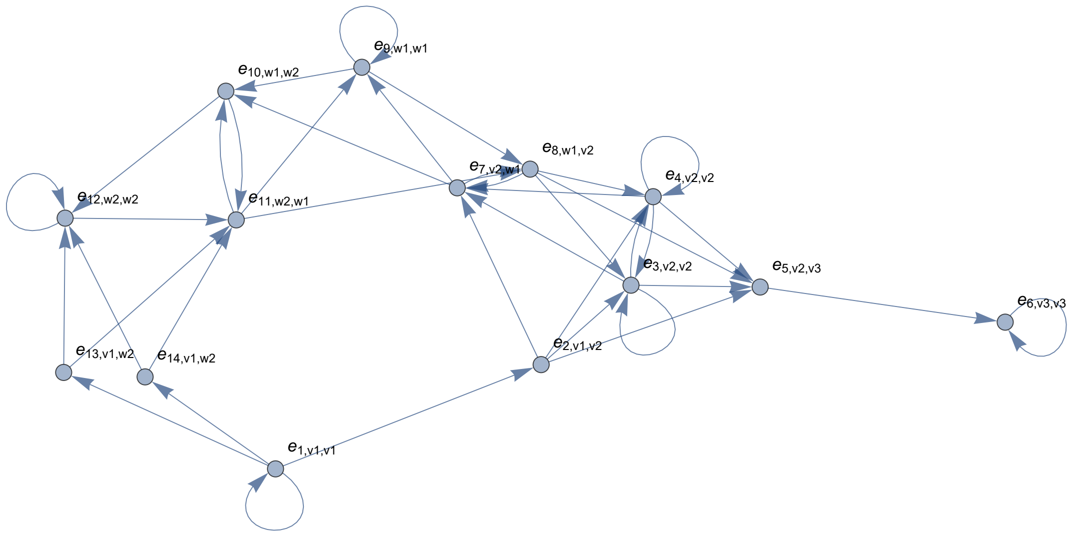
{2., 1., 1., 0., 0., 0.}

---

Graph $D_1^{(P)}$




Line graph:




Line adjacency matrix:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
$$

Line adjacency spectrum:

$$\{\text{Root}[1 + 3 \#1 - 4 \#1^2 + \#1^3 \&, 3], \text{Root}[1 + 3 \#1 - 4 \#1^2 + \#1^3 \&, 2], 1, 1, \text{Root}[1 + 3 \#1 - 4 \#1^2 + \#1^3 \&, 1], 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

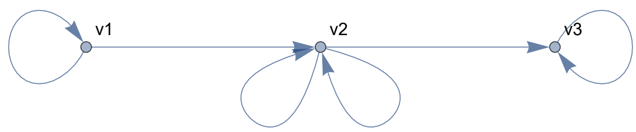
{2.80194, 1.44504, 1., 1., -0.24698, 0., 0., 0., 0., 0., 0., 0., 0., 0.}

## Hermitian adjacency spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Hermitian adjacency spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D2"];
hermitianAdjacencySpectrumS[d2]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(P)"];
hermitianAdjacencySpectrumS[d2P]
% // N
```

```
----------------------------------------------------------------------

----------------------------------------------------------------------

Hermitian adjacency spectra

----------------------------------------------------------------------

----------------------------------------------------------------------
```

Graph $D_2$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & 0 \\ -i & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

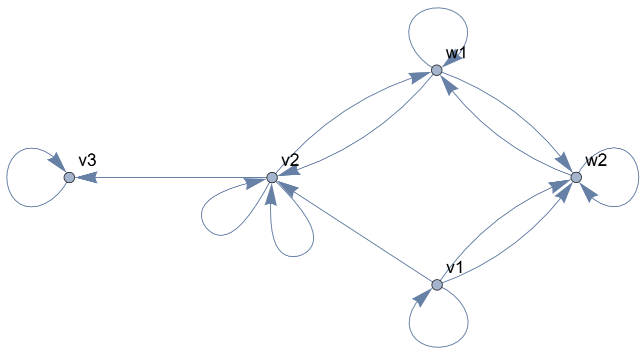
Hermitian adjacency spectrum:

$\{1+\sqrt{2}, 1, 1-\sqrt{2}\}$

{2.41421, 1., -0.414214}

----------------------------------------------------------------------

Graph $D_1^{(P)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & i & 0 & 0 & i \\ -i & 1 & i & 1 & 0 \\ 0 & -i & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ -i & 0 & 0 & 1 & 1 \end{pmatrix}$$

Hermitian adjacency spectrum:

$$\left\{\frac{1}{2}\left(2+\sqrt{2\left(5+\sqrt{17}\right)}\right), \frac{1}{2}\left(2+\sqrt{2\left(5-\sqrt{17}\right)}\right), \frac{1}{2}\left(2-\sqrt{2\left(5+\sqrt{17}\right)}\right), 1, \frac{1}{2}\left(2-\sqrt{2\left(5-\sqrt{17}\right)}\right)\right\}$$

{3.13578, 1.66215, -1.13578, 1., 0.337847}

# Skew adjacency spectra

```
Print[
  "----------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------"];
Print["Skew adjacency spectra"];
Print[
  "----------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------"];
Print["Graph D2"];
skewAdjacencySpecS[d2]
% // N
Print[
  "----------------------------------------------------------------------"];
Print["Graph D1(P)"];
skewAdjacencySpecS[d2P]
% // N
```

-----------------------------------------------------------------------

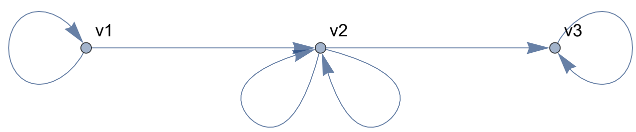
-----------------------------------------------------------------------

Skew adjacency spectra

-----------------------------------------------------------------------

-----------------------------------------------------------------------

Graph $D_2$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

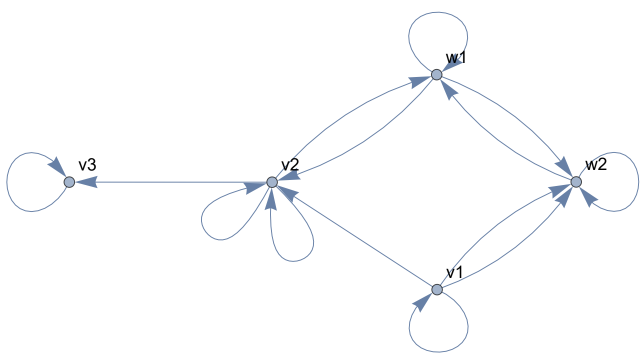
Skew adjacency spectrum:

$\{i\sqrt{2}, -i\sqrt{2}, 0\}$

$\{0. + 1.41421\,i, 0. - 1.41421\,i, 0.\}$

-----------------------------------------------------------------------

Graph $D_1^{(P)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Skew adjacency spectrum:

$\{i\sqrt{3+\sqrt{5}}, -i\sqrt{3+\sqrt{5}}, i\sqrt{3-\sqrt{5}}, -i\sqrt{3-\sqrt{5}}, 0\}$

{0. + 2.28825 i, 0. - 2.28825 i, 0. + 0.874032 i, 0. - 0.874032 i, 0.}

## Binary skew adjacency spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Binary skew adjacency spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D2"];
skewAdjacencySpecBinaryS[d2]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D1(P)"];
skewAdjacencySpecBinaryS[d2P]
% // N
```

\-----------------------------------------------------------------------------

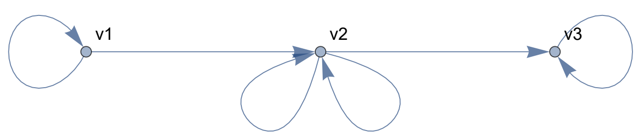
\-----------------------------------------------------------------------------

Binary skew adjacency spectra

\-----------------------------------------------------------------------------

\-----------------------------------------------------------------------------

Graph $D_2$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

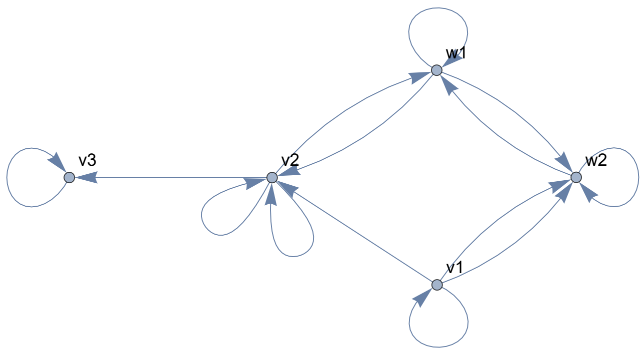
Binary skew adjacency spectrum:

$\{i\sqrt{2}, -i\sqrt{2}, 0\}$

{0. + 1.41421 i, 0. - 1.41421 i, 0.}

---------------------------------------------------------------------------

Graph $D_1^{(P)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency spectrum:

$$\left\{ i \sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)},\ -i \sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)},\ i \sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)},\ -i \sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)},\ 0 \right\}$$

{0. + 1.61803 i, 0. − 1.61803 i, 0. + 0.618034 i, 0. − 0.618034 i, 0.}

## Skew Laplace spectra

```
Print[
  "---------------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------------"];
Print["Skew Laplace spectra"];
Print[
  "---------------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------------"];
Print["Graph D2"];
skewLaplaceSpecS[d2]
% // N
Print[
  "---------------------------------------------------------------------------"];
Print["Graph D1(P)"];
skewLaplaceSpecS[d2P]
% // N
```

------------------------------------------------------------------------

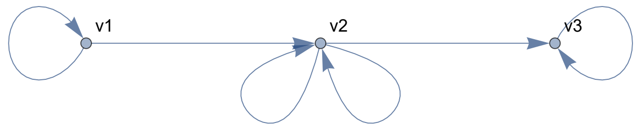
------------------------------------------------------------------------

Skew Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_2$




Skew Laplacian matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

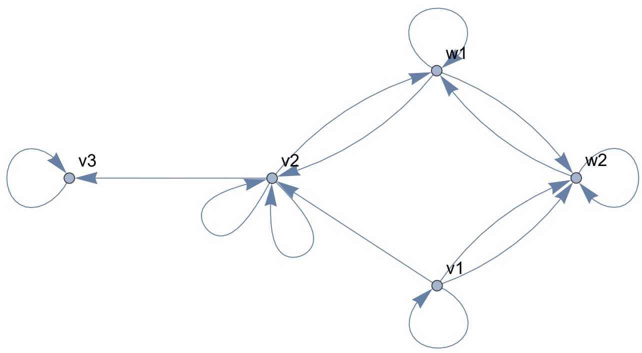
Skew Laplace spectrum

$\{i, -i, 0\}$

$\{0. + 1.\ i, 0. - 1.\ i, 0.\}$

------------------------------------------------------------------------

Graph $D_1^{(P)}$




Skew Laplacian matrix

$$\begin{pmatrix} 3 & -1 & 0 & 0 & -2 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Skew Laplace spectrum

$\{-1, 1, 0, 0, 0\}$

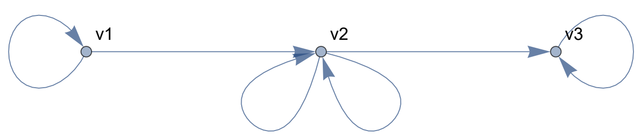
$\{-1., 1., 0., 0., 0.\}$

# Binary skew Laplace spectra

```
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Binary skew Laplace spectra"];
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D2"];
skewLaplaceSpecBinaryS[d2]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D1(P)"];
skewLaplaceSpecBinaryS[d2P]
% // N
```

```
-------------------------------------------------------------------------------

-------------------------------------------------------------------------------

Binary skew Laplace spectra

-------------------------------------------------------------------------------

-------------------------------------------------------------------------------
```

Graph $D_2$




Binary skew Laplacian matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

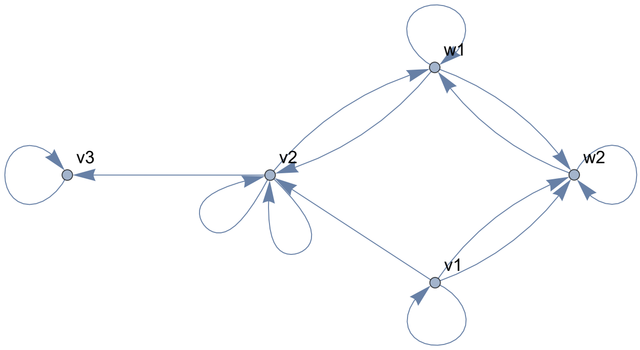
Binary skew Laplace spectrum

{i, -i, 0}

{0. + 1. i, 0. - 1. i, 0.}

------------------------------------------------------------------------

Graph $D_1^{(P)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Binary skew Laplace spectrum

{0, 0, 0, 0, 0}

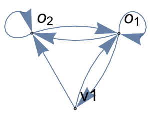
{0., 0., 0., 0., 0.}

# Example 5.3 (of arXiv:2010.10769 [math.CO] v1)

## Definitions of the graphs

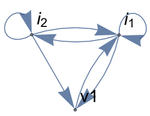
```
(* Define the graph D3 *)
Print["Graph D3"];
d3 = Graph[{v1, v2}, {v1 ⟷ v2, v2 ⟷ v1, v2 ⟷ v2, v2 ⟷ v2}, VertexLabels → "Name"]

(* Define the graph D3^(O) *)
(* computed using d3O=moveO[d1,o] *)
Print["Graph D3^(O)"];
```

d3O =

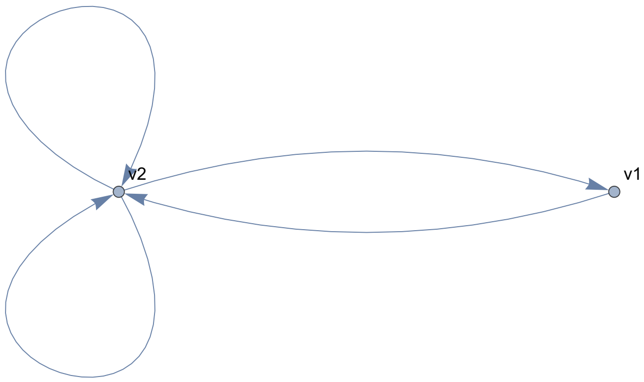



```
(* Define the graph D3^(I) *)
(* computed using d3I=moveI[d3,i] *)
Print["Graph D3^(I)"];
```

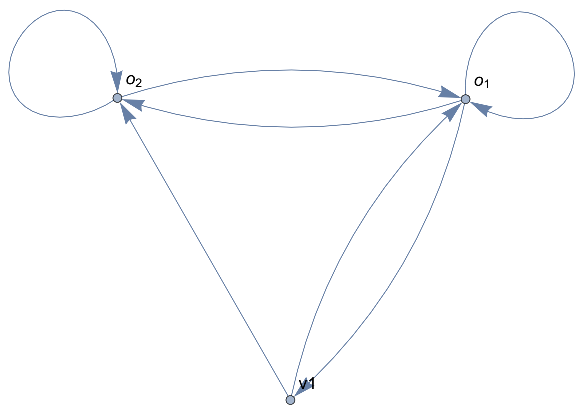
d3I =



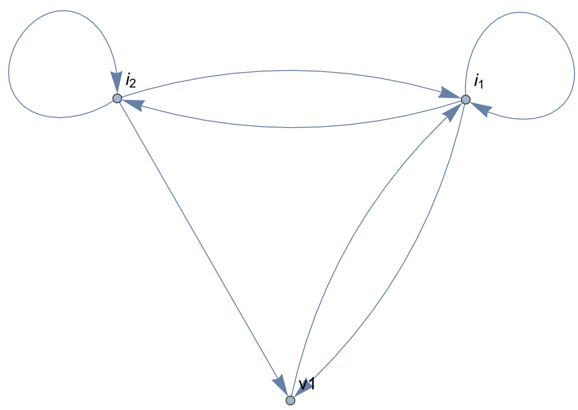

Graph $D_3$




Graph $D_3^{(O)}$




Graph $D_3^{(I)}$

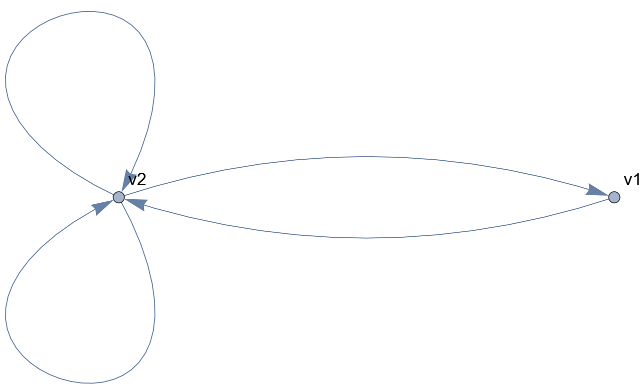


## Binary adjacency spectra

```
Print[
  "----------------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------------"];
Print["Binary adjacency spectra"];
Print[
  "----------------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D3"];
adjacencySpecBinaryS[d3]
% // N
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D3(O)"];
adjacencySpecBinaryS[d3O]
% // N
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D3(I)"];
adjacencySpecBinaryS[d3I]
% // N
```

```
----------------------------------------------------------------------------------

----------------------------------------------------------------------------------

Binary adjacency spectra

----------------------------------------------------------------------------------

----------------------------------------------------------------------------------
```

Graph $D_3$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

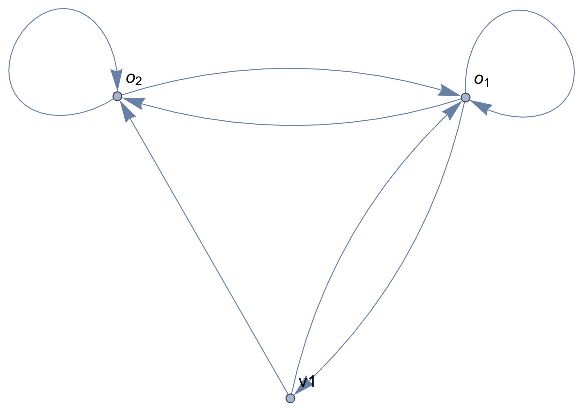
Binary adjacency spectrum:

$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right\}$

{1.61803, -0.618034}

-----------------------------------------------------------------------------

Graph $D_3^{(O)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

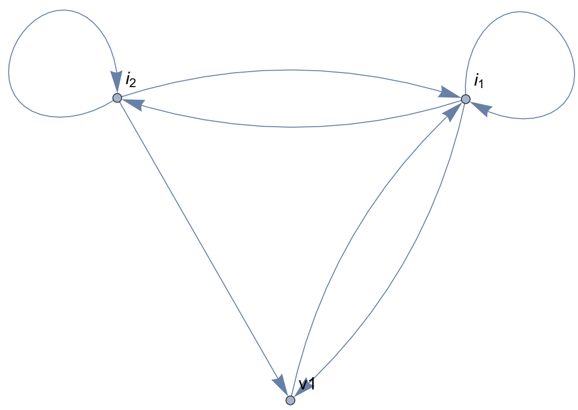
Binary adjacency spectrum:

$\left\{1+\sqrt{2}, 1-\sqrt{2}, 0\right\}$

{2.41421, -0.414214, 0.}

-----------------------------------------------------------------------------

Graph $D_3^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Binary adjacency spectrum:

$\left\{1+\sqrt{2}, 1-\sqrt{2}, 0\right\}$

```
{2.41421, -0.414214, 0.}
```

## Normalized Laplace spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Normalized Laplace spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D3"];
normalizedLaplaceSpecS[d3]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D3^(O)"];
normalizedLaplaceSpecS[d3O]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D3^(I)"];
normalizedLaplaceSpecS[d3I]
% // N
```

------------------------------------------------------------------------

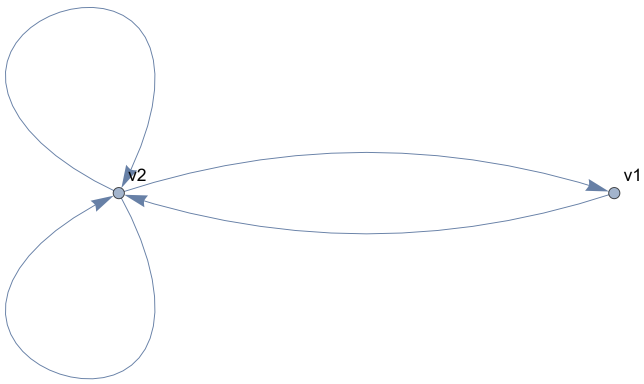
------------------------------------------------------------------------

Normalized Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_3$




Transition probability matrix:

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Perron−Frobenius vector:

$$\left\{\frac{1}{4}, \frac{3}{4}\right\}$$

Perron−Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}$$

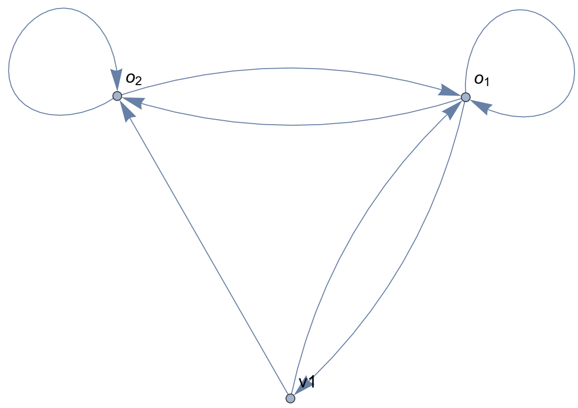
Normalized Laplace spectrum:

$$\left\{\frac{4}{3}, 0\right\}$$

{1.33333, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(0)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{\sqrt{3}}{4} & -\frac{1}{4\sqrt{3}} \\ -\frac{\sqrt{3}}{4} & \frac{2}{3} & -\frac{5}{12} \\ -\frac{1}{4\sqrt{3}} & -\frac{5}{12} & \frac{1}{2} \end{pmatrix}$$

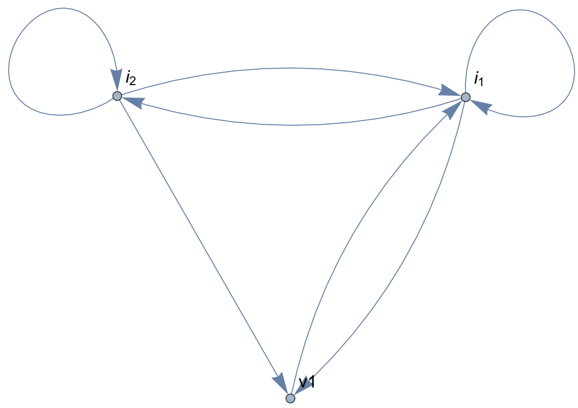
Normalized Laplace spectrum:

$$\left\{\frac{1}{12}\left(13+2\sqrt{2}\right), \frac{1}{12}\left(13-2\sqrt{2}\right), 0\right\}$$

{1.31904, 0.847631, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(I)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} 1 & \frac{1}{2}\left(-\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3}\right) & -\frac{1}{6} \\ \frac{1}{2}\left(-\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3}\right) & \frac{2}{3} & \frac{1}{2}\left(-\frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3}\right) \\ -\frac{1}{6} & \frac{1}{2}\left(-\frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3}\right) & \frac{2}{3} \end{pmatrix}$$

Normalized Laplace spectrum:

$$\left\{\frac{1}{6}\left(7+\sqrt{3}\right), \frac{1}{6}\left(7-\sqrt{3}\right), 0\right\}$$

{1.45534, 0.877992, 0.}

# Combinatorial Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Combinatorial Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D3"];
combinatorialLaplaceSpecS[d3]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D3^(O)"];
combinatorialLaplaceSpecS[d3O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D3^(I)"];
combinatorialLaplaceSpecS[d3I]
% // N
```

------------------------------------------------------------------------

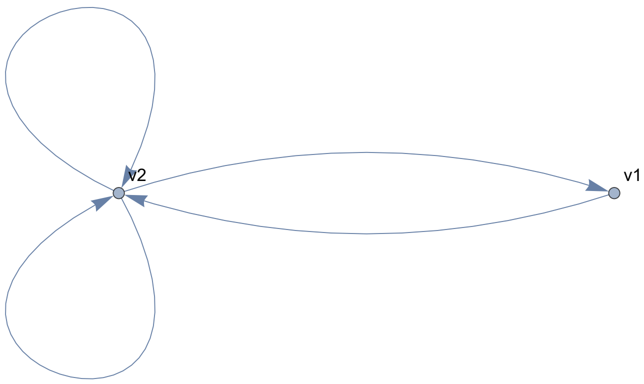
------------------------------------------------------------------------

Combinatorial Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_3$




Transition probability matrix:

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{4}, \frac{3}{4}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

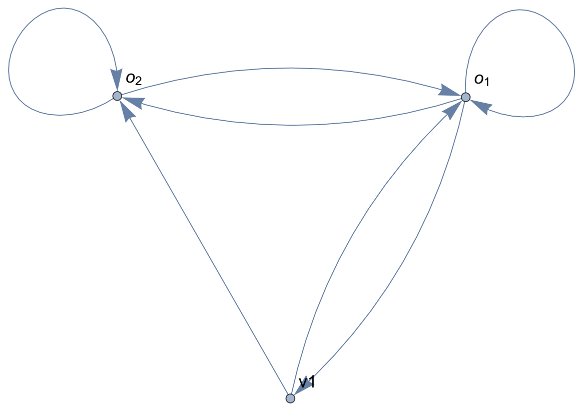
Combinatorial Laplace spectrum:

$$\left\{\frac{1}{2}, 0\right\}$$

{0.5, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(0)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{7} & -\frac{3}{28} & -\frac{1}{28} \\ -\frac{3}{28} & \frac{2}{7} & -\frac{5}{28} \\ -\frac{1}{28} & -\frac{5}{28} & \frac{3}{14} \end{pmatrix}$$

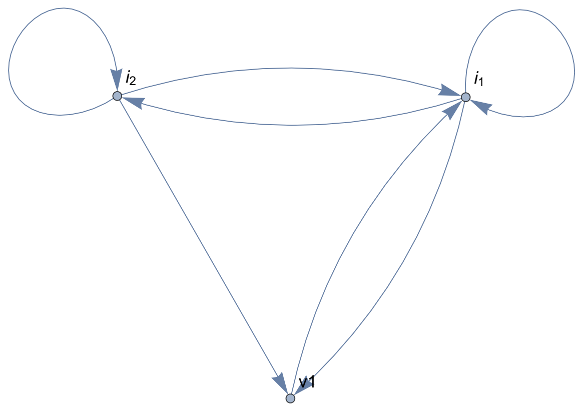
Combinatorial Laplace spectrum:

$$\left\{\frac{1}{28}\left(9+2\sqrt{3}\right), \frac{1}{28}\left(9-2\sqrt{3}\right), 0\right\}$$

{0.445146, 0.197711, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(I)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{4} & -\frac{5}{24} & -\frac{1}{24} \\ -\frac{5}{24} & \frac{1}{3} & -\frac{1}{8} \\ -\frac{1}{24} & -\frac{1}{8} & \frac{1}{6} \end{pmatrix}$$

Combinatorial Laplace spectrum:

$$\left\{\frac{1}{24}\left(9+2\sqrt{3}\right), \frac{1}{24}\left(9-2\sqrt{3}\right), 0\right\}$$

{0.519338, 0.230662, 0.}

## Binary normalized Laplace spectra

```
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Binary normalized Laplace spectra"];
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Graph D3"];
normalizedLaplaceSpecBinaryS[d3]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D3^(O)"];
normalizedLaplaceSpecBinaryS[d3O]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D3^(I)"];
normalizedLaplaceSpecBinaryS[d3I]
% // N
```

---------------------------------------------------------------------------

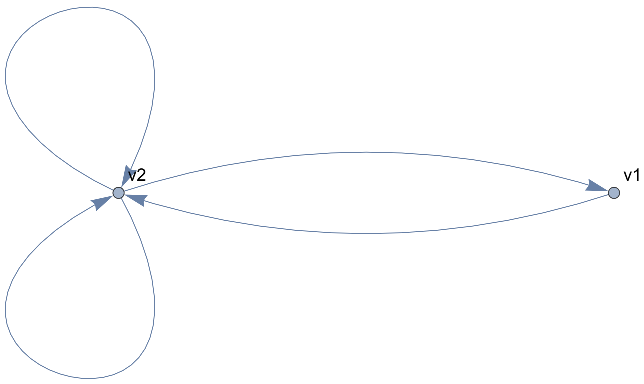
---------------------------------------------------------------------------

Binary normalized Laplace spectra

---------------------------------------------------------------------------

---------------------------------------------------------------------------

Graph $D_3$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{3}, \frac{2}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

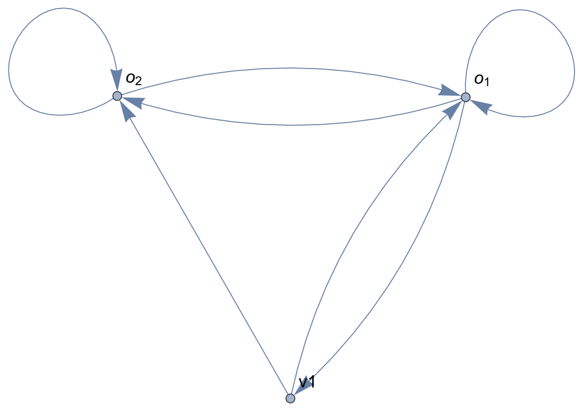
Binary normalized Laplace spectrum:

$$\left\{\frac{3}{2}, 0\right\}$$

{1.5, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(0)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Perron−Frobenius vector:

$$\left\{\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right\}$$

Perron−Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{\sqrt{3}}{4} & -\frac{1}{4\sqrt{3}} \\ -\frac{\sqrt{3}}{4} & \frac{2}{3} & -\frac{5}{12} \\ -\frac{1}{4\sqrt{3}} & -\frac{5}{12} & \frac{1}{2} \end{pmatrix}$$

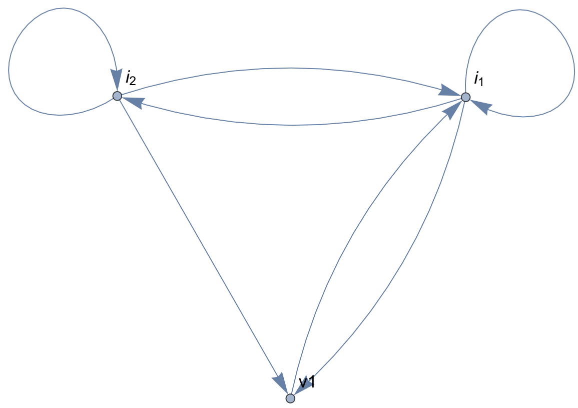
Binary normalized Laplace spectrum:

$$\left\{\frac{1}{12}\left(13 + 2\sqrt{2}\right), \frac{1}{12}\left(13 - 2\sqrt{2}\right), 0\right\}$$

{1.31904, 0.847631, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(I)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} 1 & \frac{1}{2}\left(-\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3}\right) & -\frac{1}{6} \\ \frac{1}{2}\left(-\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3}\right) & \frac{2}{3} & \frac{1}{2}\left(-\frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3}\right) \\ -\frac{1}{6} & \frac{1}{2}\left(-\frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3}\right) & \frac{2}{3} \end{pmatrix}$$

Binary normalized Laplace spectrum:

$$\left\{\frac{1}{6}\left(7+\sqrt{3}\right), \frac{1}{6}\left(7-\sqrt{3}\right), 0\right\}$$

{1.45534, 0.877992, 0.}

## Binary combinatorial Laplace spectra

```
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Binary combinatorial Laplace spectra"];
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D3"];
combinatorialLaplaceSpecBinaryS[d3]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D3^(O)"];
combinatorialLaplaceSpecBinaryS[d3O]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D3^(I)"];
combinatorialLaplaceSpecBinaryS[d3I]
% // N
```

----------------------------------------------------------------------------

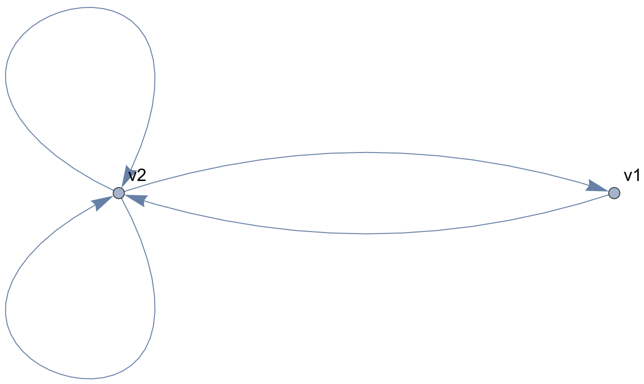
----------------------------------------------------------------------------

Binary combinatorial Laplace spectra

----------------------------------------------------------------------------

----------------------------------------------------------------------------

Graph $D_3$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{3}, \frac{2}{3}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

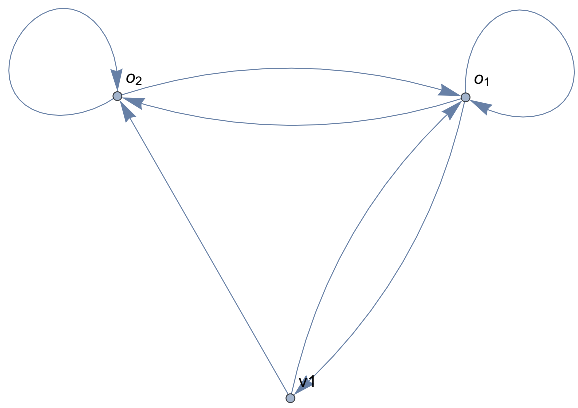
Binary combinatorial Laplace spectrum:

$$\left\{\frac{2}{3}, 0\right\}$$

{0.666667, 0.}

---------------------------------------------------------------------------

Graph $D_3^{(0)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{7} & -\frac{3}{28} & -\frac{1}{28} \\ -\frac{3}{28} & \frac{2}{7} & -\frac{5}{28} \\ -\frac{1}{28} & -\frac{5}{28} & \frac{3}{14} \end{pmatrix}$$

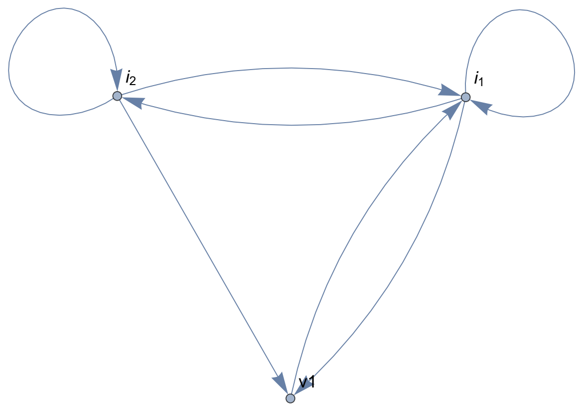
Binary combinatorial Laplace spectrum:

$$\left\{\frac{1}{28}\left(9+2\sqrt{3}\right), \frac{1}{28}\left(9-2\sqrt{3}\right), 0\right\}$$

{0.445146, 0.197711, 0.}

----------------------------------------------------------------------------

Graph $D_3^{(I)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{4} & -\frac{5}{24} & -\frac{1}{24} \\ -\frac{5}{24} & \frac{1}{3} & -\frac{1}{8} \\ -\frac{1}{24} & -\frac{1}{8} & \frac{1}{6} \end{pmatrix}$$

Binary combinatorial Laplace spectrum:

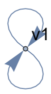
$$\left\{\frac{1}{24}\left(9+2\sqrt{3}\right), \frac{1}{24}\left(9-2\sqrt{3}\right), 0\right\}$$

$\{0.519338, 0.230662, 0.\}$

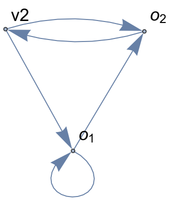
# Example 5.4 (of arXiv:2010.10769 [math.CO] v1)

## Definitions of the graphs

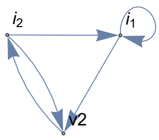
```
(* Define the graph D4 *)
Print["Graph D4"];
d4 = Graph[{v1, v2}, {v1 ↔ v1, v1 ↔ v2, v2 ↔ v1}, VertexLabels → "Name"]

(* Define the graph D4^(R) *)
(* computed using d4R=moveR[d4] *)
Print["Graph D4^(R)"];
```

d4R = 

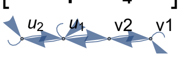

```
(* Define the graph D4^(O) *)
(* computed using d4O=moveO[d4,o] *)
Print["Graph D4^(O)"];
```

d4O = 

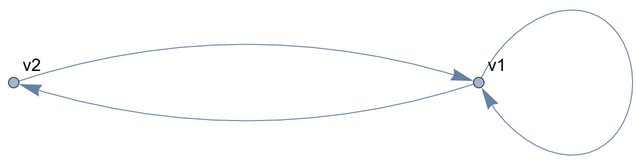

```
(* Define the graph D4^(I) *)
(* computed using d4I=moveI[d4,i] *)
Print["Graph D4^(I)"];
```

d4I = 

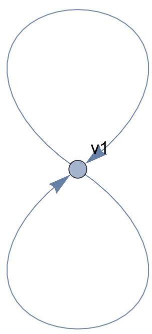

```
(* Define the graph D4^(C) *)
(* computed using d4C=moveC[d4,u] *)
Print["Graph D4^(C)"];
```

d4C = 

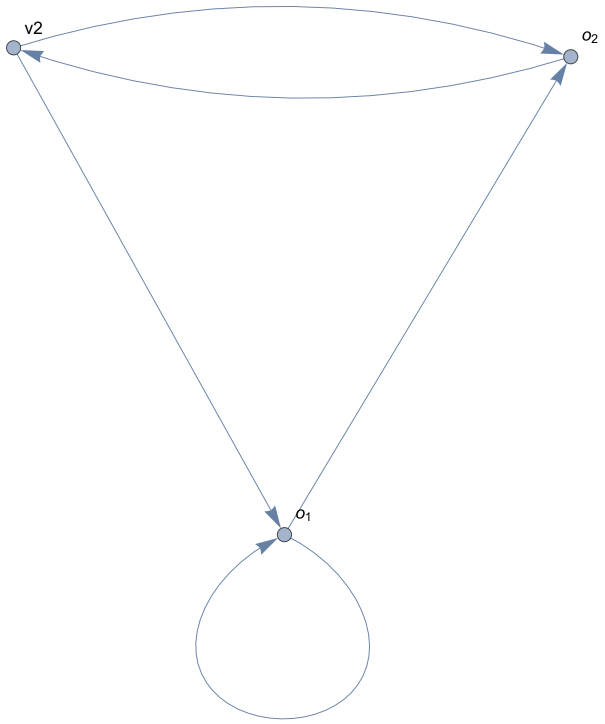

Graph $D_4$



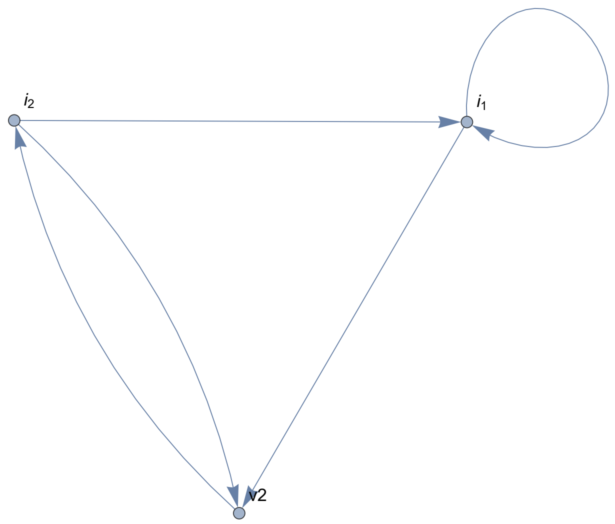

Graph $D_4^{(R)}$



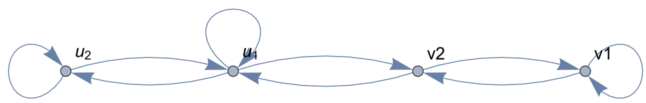

Graph $D_4^{(O)}$




Graph $D_4^{(I)}$




Graph $D_4^{(C)}$



## Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4"];
laplaceSpecS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
laplaceSpecS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
laplaceSpecS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
laplaceSpecS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
laplaceSpecS[d4C]
% // N
```

------------------------------------------------------------------------

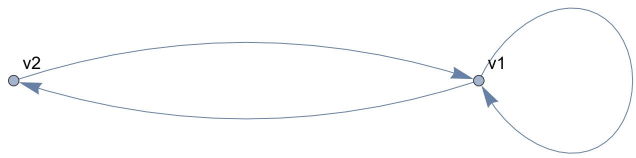
------------------------------------------------------------------------

Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_4$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

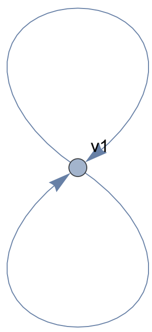
Laplace spectrum:

{4, 0}

{4., 0.}

------------------------------------------------------------------------------

Graph $D_4^{(R)}$




Incidence matrix:

$\begin{pmatrix} 0 & 0 \end{pmatrix}$

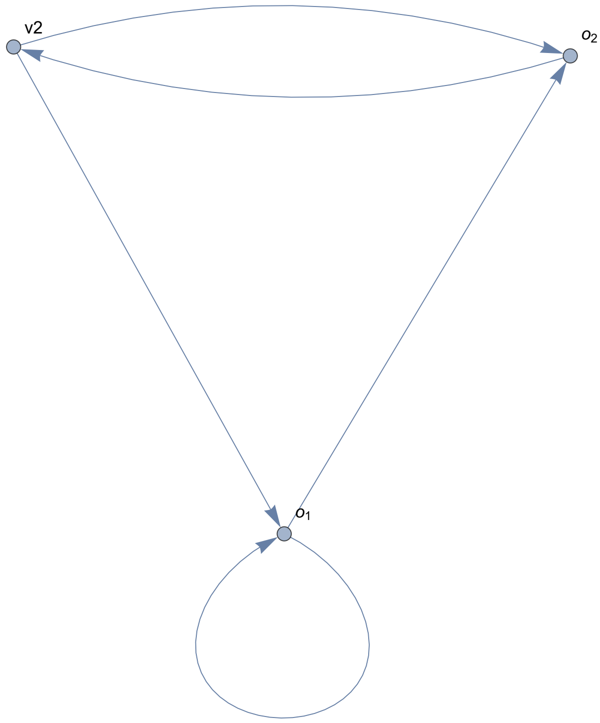
Laplace spectrum:

{0}

{0.}

------------------------------------------------------------------------

Graph $D_4^{(0)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix}$$

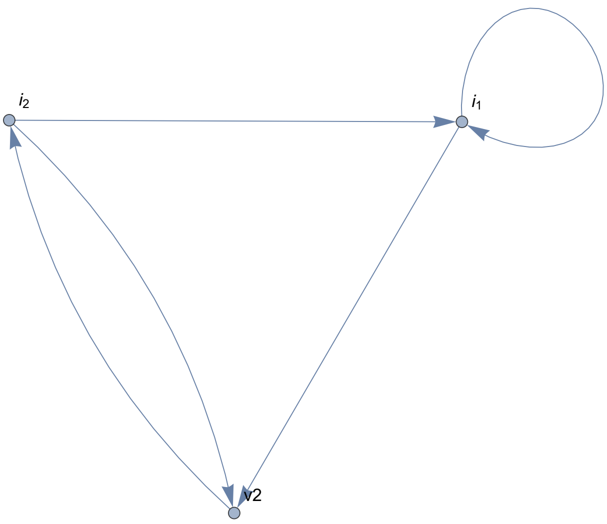
Laplace spectrum:

{5, 3, 0}

{5., 3., 0.}

----------------------------------------------------------------------

Graph $D_4^{(I)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

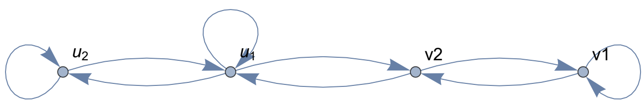
Laplace spectrum:

{5, 3, 0}

{5., 3., 0.}

----------------------------------------------------------------------

Graph $D_4^{(C)}$




Incidence matrix:

$$\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Laplace spectrum:

$\{2\left(2+\sqrt{2}\right), 4, 2\left(2-\sqrt{2}\right), 0\}$

{6.82843, 4., 1.17157, 0.}

## Adjacency spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Adjacency spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4"];
adjacencySpecS[d4]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(R)"];
adjacencySpecS[d4R]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(O)"];
adjacencySpecS[d4O]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(I)"];
adjacencySpecS[d4I]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(C)"];
adjacencySpecS[d4C]
% // N
```

---------------------------------------------------------------------------

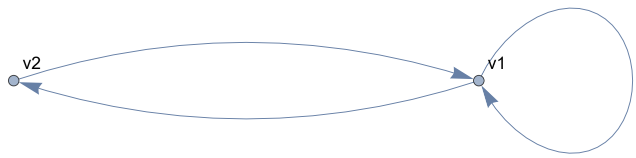
---------------------------------------------------------------------------

Adjacency spectra

---------------------------------------------------------------------------

---------------------------------------------------------------------------

Graph $D_4$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

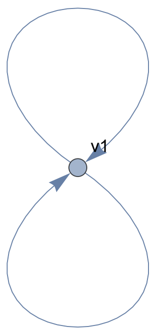
Adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right\}$$

{1.61803, -0.618034}

------------------------------------------------------------------------

Graph $D_4^{(R)}$




Adjacency matrix:

( 2 )

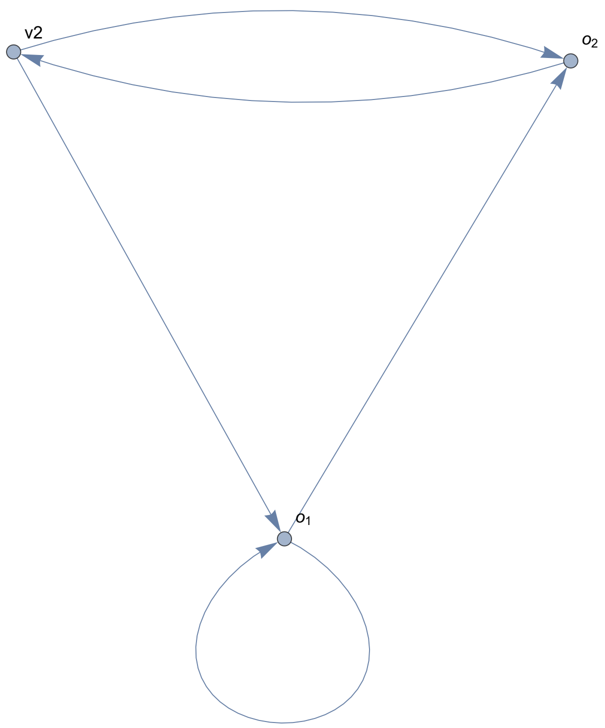
Adjacency spectrum:

{2}

{2.}

------------------------------------------------------------------

Graph $D_4^{(0)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

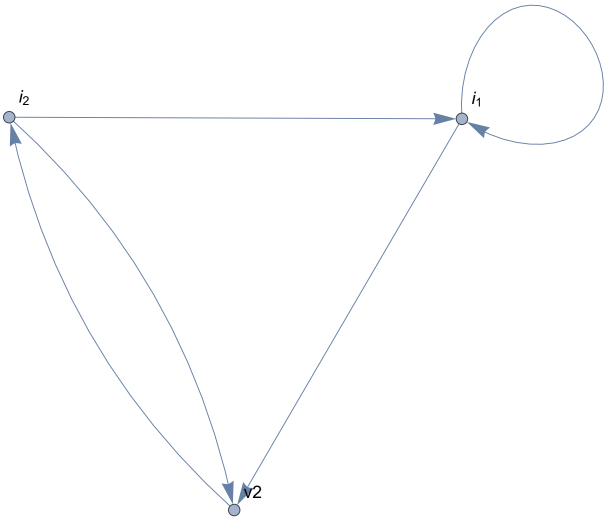
Adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0\right\}$$

{1.61803, -0.618034, 0.}

---

Graph $D_4^{(I)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

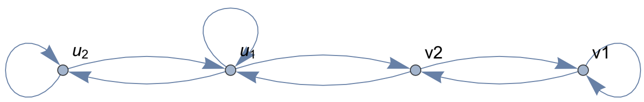
Adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0\right\}$$

{1.61803, -0.618034, 0.}

---

Graph $D_4^{(C)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Adjacency spectrum:

{Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 4], Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 3],
Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 1], Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 2]}

{2.35567, 1.47726, -1.09529, 0.26236}

## Line adjacency spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Line adjacency spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4"];
lineAdjacencySpecS[d4]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(R)"];
lineAdjacencySpecS[d4R]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(O)"];
lineAdjacencySpecS[d4O]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(I)"];
lineAdjacencySpecS[d4I]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(C)"];
lineAdjacencySpecS[d4C]
% // N
```

----------------------------------------------------------------------------

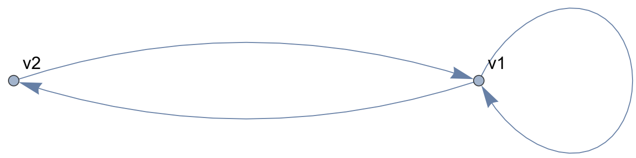
----------------------------------------------------------------------------

Line adjacency spectra

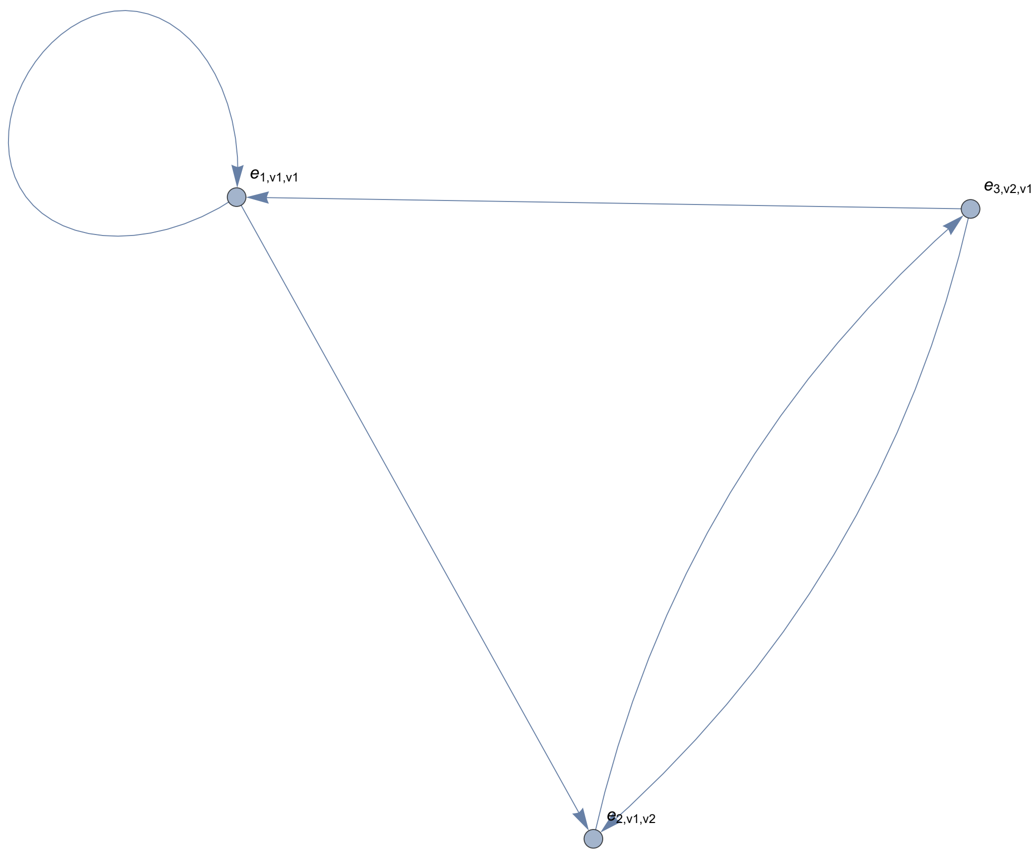
----------------------------------------------------------------------------

----------------------------------------------------------------------------

Graph $D_4$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

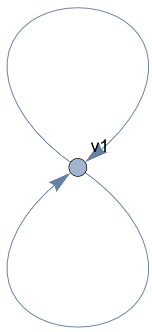
Line adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0\right\}$$

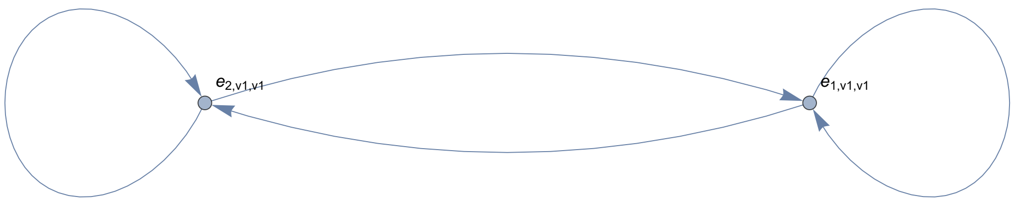
{1.61803, -0.618034, 0.}

---

Graph $D_4^{(R)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

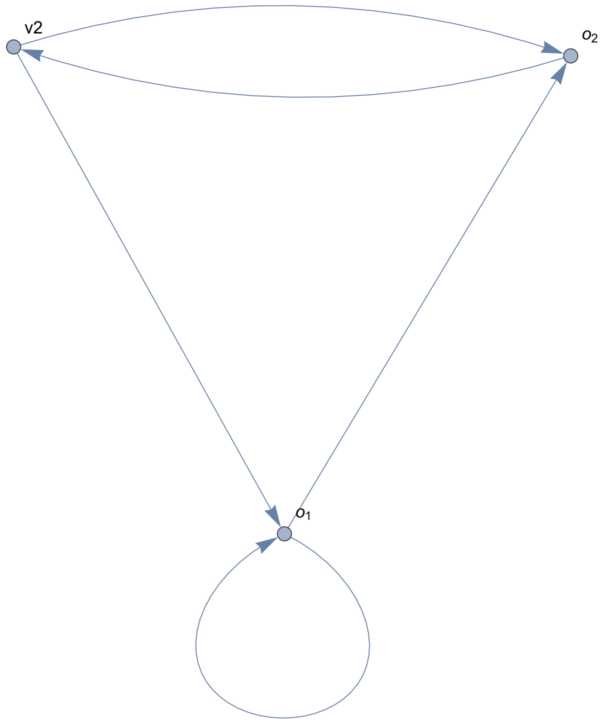
Line adjacency spectrum:

{2, 0}

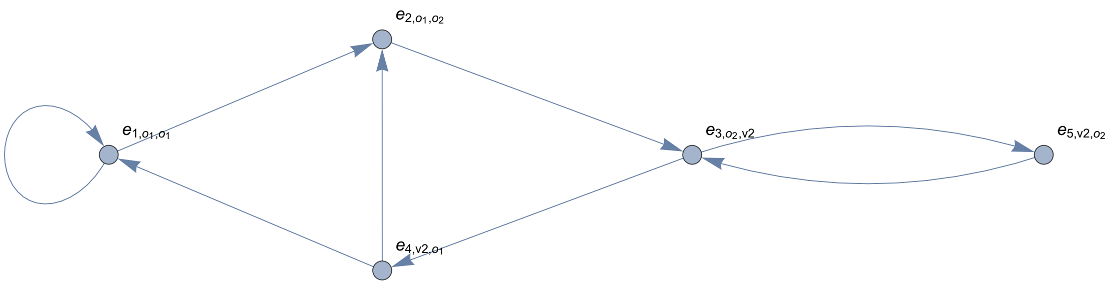
{2., 0.}

---

Graph $D_4^{(0)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

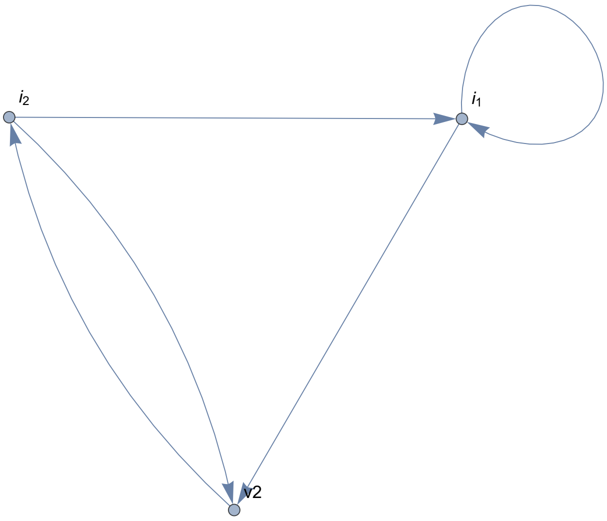
Line adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0, 0, 0\right\}$$

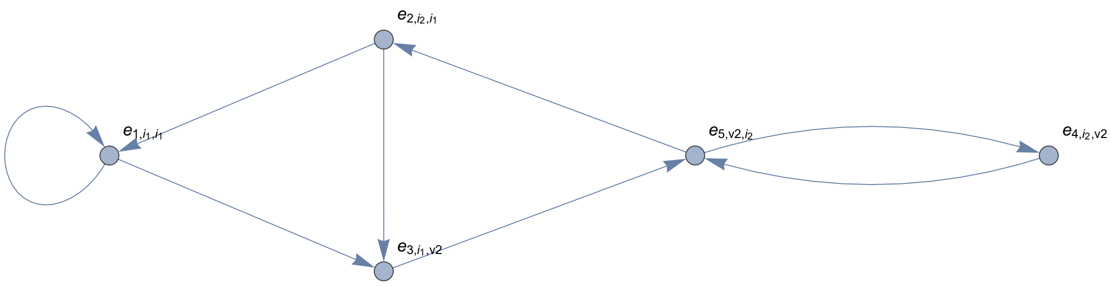
{1.61803, -0.618034, 0., 0., 0.}

---------------------------------------------------------------------

Graph $D_4^{(I)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

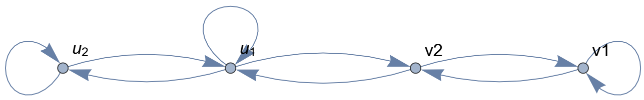
Line adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0, 0, 0\right\}$$

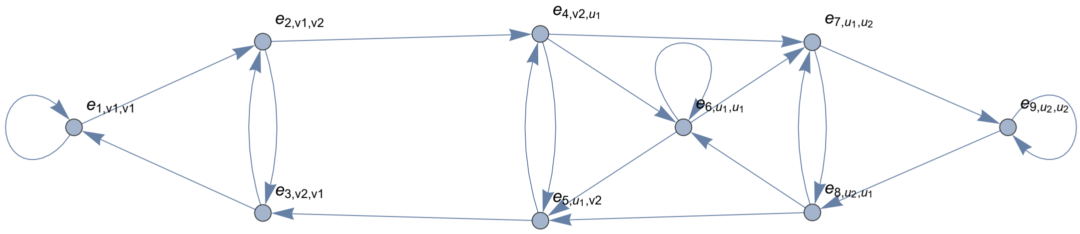
$\{1.61803, -0.618034, 0., 0., 0.\}$

----------------------------------------------------------------------

Graph $D_4^{(C)}$




Line graph:




Line adjacency matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Line adjacency spectrum:

$\{\text{Root}[-1 + 4 \#1 - 3 \#1^3 + \#1^4 \&, 4], \text{Root}[-1 + 4 \#1 - 3 \#1^3 + \#1^4 \&, 3],$
$\text{Root}[-1 + 4 \#1 - 3 \#1^3 + \#1^4 \&, 1], \text{Root}[-1 + 4 \#1 - 3 \#1^3 + \#1^4 \&, 2], 0, 0, 0, 0, 0\}$

{2.35567, 1.47726, -1.09529, 0.26236, 0., 0., 0., 0., 0.}

## Binary adjacency spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Binary adjacency spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4"];
adjacencySpecBinaryS[d4]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
adjacencySpecBinaryS[d4R]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
adjacencySpecBinaryS[d4O]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
adjacencySpecBinaryS[d4I]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
adjacencySpecBinaryS[d4C]
% // N
```

-----------------------------------------------------------------------------

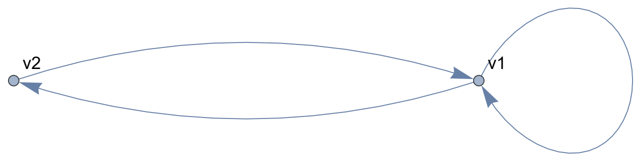
-----------------------------------------------------------------------------

Binary adjacency spectra

-----------------------------------------------------------------------------

-----------------------------------------------------------------------------

Graph $D_4$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

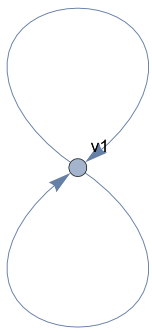
Binary adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right),\ \frac{1}{2}\left(1-\sqrt{5}\right)\right\}$$

$\{1.61803, -0.618034\}$

----------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary adjacency matrix:

$\begin{pmatrix} 1 \end{pmatrix}$

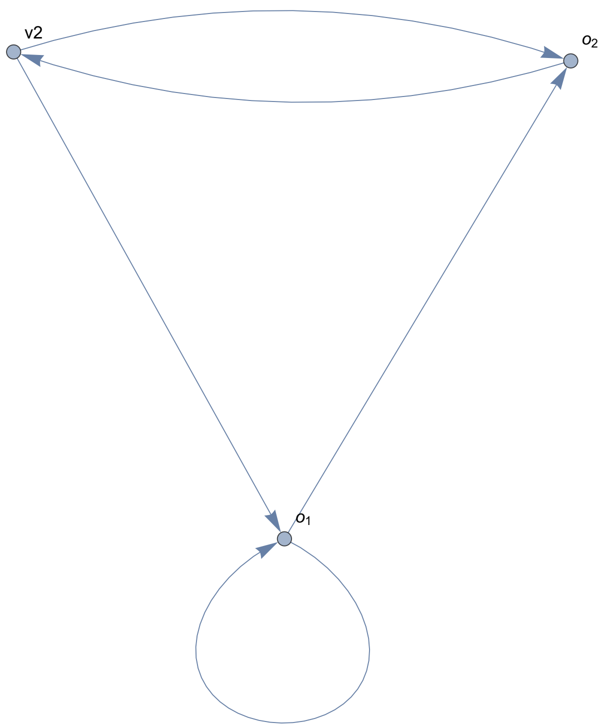
Binary adjacency spectrum:

{1}

{1.}

---------------------------------------------------------------------

Graph $D_4^{(0)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

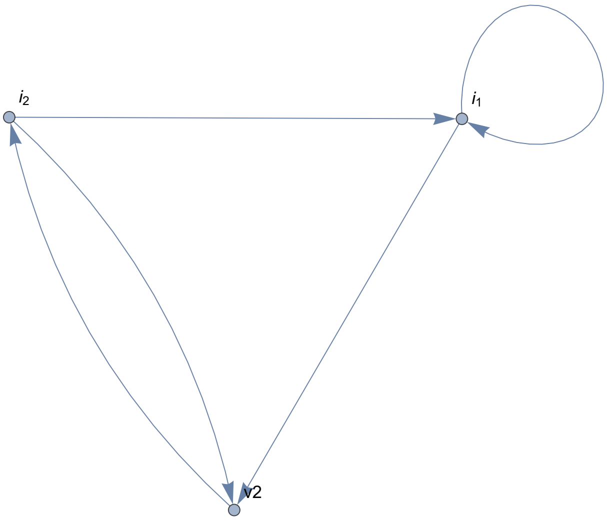
Binary adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0\right\}$$

$\{1.61803, -0.618034, 0.\}$

---

Graph $D_4^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

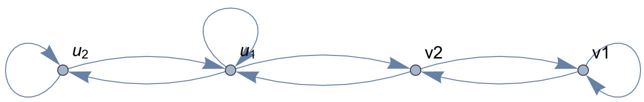
Binary adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 0\right\}$$

{1.61803, -0.618034, 0.}

---

Graph $D_4^{(C)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Binary adjacency spectrum:

{Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 4], Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 3],
Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 1], Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 2]}

{2.35567, 1.47726, -1.09529, 0.26236}

## Symmetric adjacency spectra

```
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Symmetric adjacency spectra"];
Print[
  "--------------------------------------------------------------------------------"];
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4"];
adjacencySpecSymmetricS[d4]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
adjacencySpecSymmetricS[d4R]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
adjacencySpecSymmetricS[d4O]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
adjacencySpecSymmetricS[d4I]
% // N
Print[
  "--------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
adjacencySpecSymmetricS[d4C]
% // N
```

---------------------------------------------------------------------------

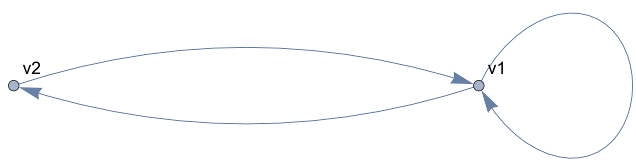
---------------------------------------------------------------------------

Symmetric adjacency spectra

---------------------------------------------------------------------------

---------------------------------------------------------------------------

Graph $D_4$




Adjacency matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

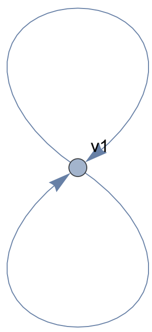
Symmetric adjacency spectrum:

$$\left\{\frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{5}\right)\right\}$$

{2.61803, 0.381966}

------------------------------------------------------------------------

Graph $D_4^{(R)}$




Adjacency matrix:

( 2 )

Adjacency matrix times its transpose:

( 4 )

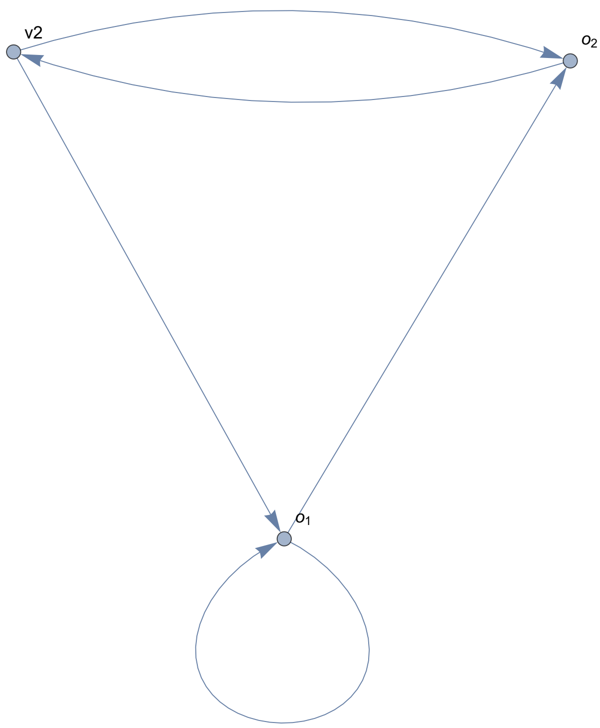
Symmetric adjacency spectrum:

{4}

{4.}

---

Graph $D_4^{(0)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

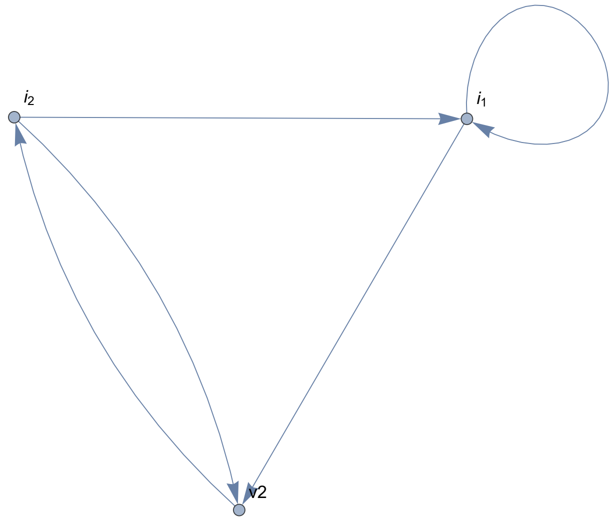
Symmetric adjacency spectrum:

{4, 1, 0}

{4., 1., 0.}

---

Graph $D_4^{(I)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

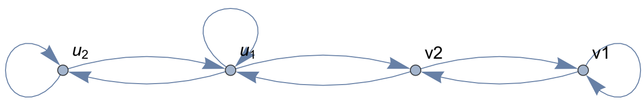
Symmetric adjacency spectrum:

{4, 1, 0}

{4., 1., 0.}

---

Graph $D_4^{(C)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Symmetric adjacency spectrum:

{Root[1 - 16 #1 + 22 #1^2 - 9 #1^3 + #1^4 &, 4], Root[1 - 16 #1 + 22 #1^2 - 9 #1^3 + #1^4 &, 3],
Root[1 - 16 #1 + 22 #1^2 - 9 #1^3 + #1^4 &, 2], Root[1 - 16 #1 + 22 #1^2 - 9 #1^3 + #1^4 &, 1]}

```
{5.5492, 2.1823, 1.19967, 0.0688326}
```

## Symmetric binary adjacency spectra

```
Print[
  "----------------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------------"];
Print["Symmetric binary adjacency spectra"];
Print[
  "----------------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D4"];
adjacencySpecSymmetricBinaryS[d4]
% // N
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D4(R)"];
adjacencySpecSymmetricBinaryS[d4R]
% // N
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D4(O)"];
adjacencySpecSymmetricBinaryS[d4O]
% // N
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D4(I)"];
adjacencySpecSymmetricBinaryS[d4I]
% // N
Print[
  "----------------------------------------------------------------------------------"];
Print["Graph D4(C)"];
adjacencySpecSymmetricBinaryS[d4C]
% // N
```

------------------------------------------------------------------------

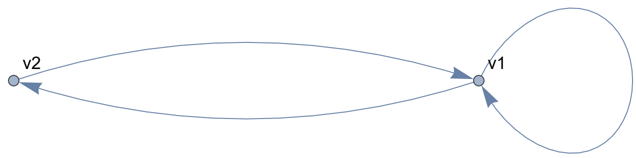
------------------------------------------------------------------------

Symmetric binary adjacency spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_4$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

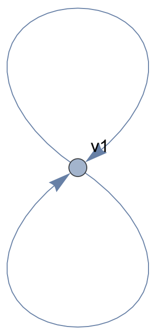
Symmetric binary adjacency spectrum:

$$\left\{\frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{5}\right)\right\}$$

{2.61803, 0.381966}

---------------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary adjacency matrix:

$\begin{pmatrix} 1 \end{pmatrix}$

Binary adjacency matrix times its transpose:

$\begin{pmatrix} 1 \end{pmatrix}$

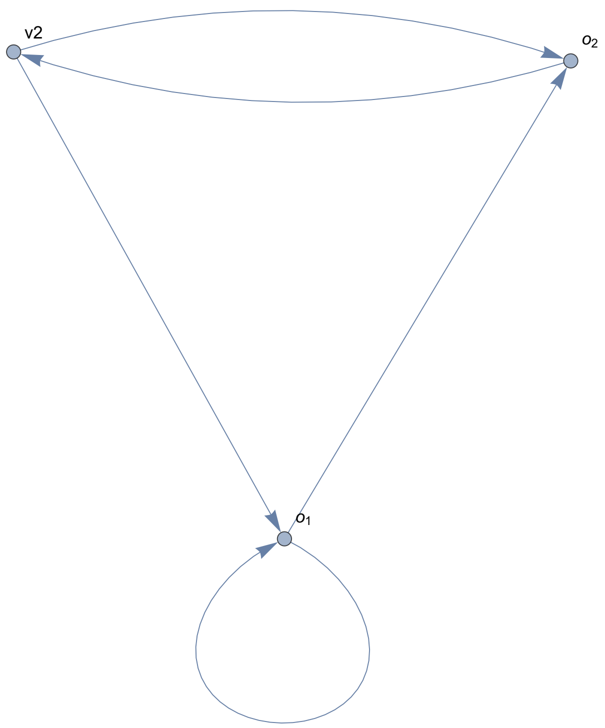
Symmetric binary adjacency spectrum:

{1}

{1.}

---------------------------------------------------------------------------

Graph $D_4^{(0)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

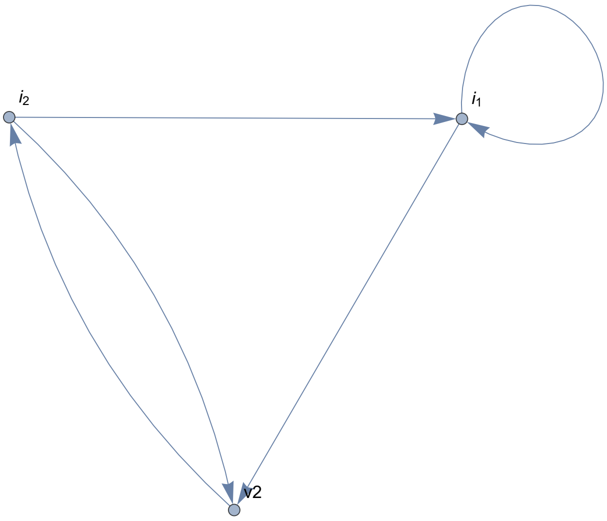
Symmetric binary adjacency spectrum:

{4, 1, 0}

{4., 1., 0.}

--------------------------------------------------------------------------

Graph $D_4^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

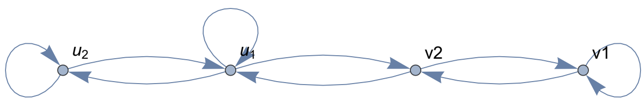
Symmetric binary adjacency spectrum:

{4, 1, 0}

{4., 1., 0.}

--------------------------------------------------------------------------

Graph $D_4^{(C)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Binary adjacency matrix times its transpose:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Symmetric binary adjacency spectrum:

$\{\mathrm{Root}[1 - 16 \#1 + 22 \#1^2 - 9 \#1^3 + \#1^4 \&, 4], \mathrm{Root}[1 - 16 \#1 + 22 \#1^2 - 9 \#1^3 + \#1^4 \&, 3],$
$\mathrm{Root}[1 - 16 \#1 + 22 \#1^2 - 9 \#1^3 + \#1^4 \&, 2], \mathrm{Root}[1 - 16 \#1 + 22 \#1^2 - 9 \#1^3 + \#1^4 \&, 1]\}$

```
{5.5492, 2.1823, 1.19967, 0.0688326}
```

## Hermitian adjacency spectra

```
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Hermitian adjacency spectra"];
Print[
  "------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------"];
Print["Graph D4"];
hermitianAdjacencySpectrumS[d4]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D4^(R)"];
hermitianAdjacencySpectrumS[d4R]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D4^(O)"];
hermitianAdjacencySpectrumS[d4O]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D4^(I)"];
hermitianAdjacencySpectrumS[d4I]
% // N
Print[
  "------------------------------------------------------------------------"];
Print["Graph D4^(C)"];
hermitianAdjacencySpectrumS[d4C]
% // N
```

---------------------------------------------------------------------

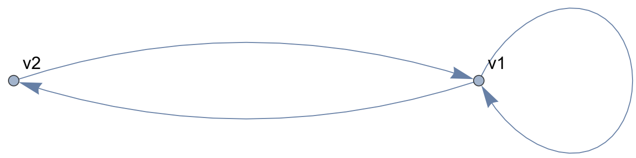
---------------------------------------------------------------------

Hermitian adjacency spectra

---------------------------------------------------------------------

---------------------------------------------------------------------

Graph $D_4$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

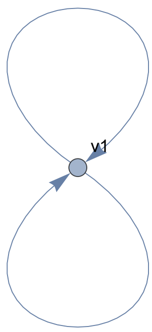
Hermitian adjacency spectrum:

$$\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right\}$$

{1.61803, -0.618034}

-----------------------------------------------------------------------------

Graph $D_4^{(R)}$




Hermitian adjacency matrix:

( 1 )

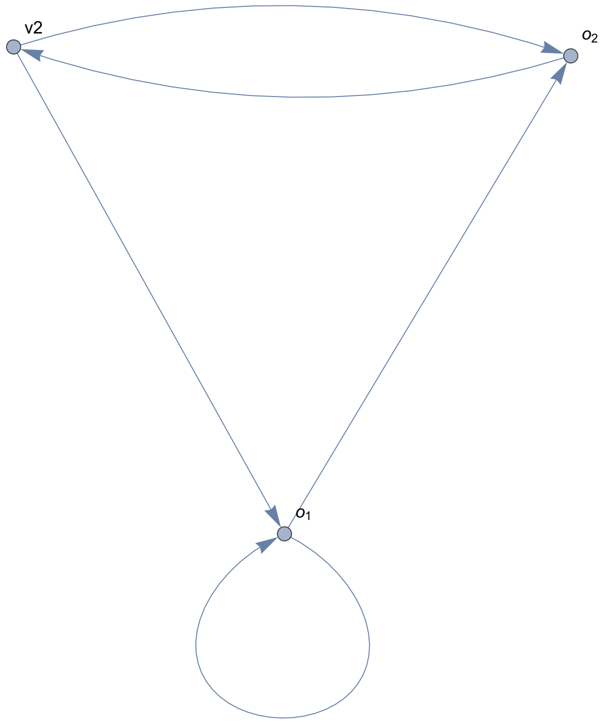
Hermitian adjacency spectrum:

{1}

{1.}

-----------------------------------------------------------------------------

Graph $D_4^{(0)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 0 & i & 1 \\ -i & 1 & i \\ 1 & -i & 0 \end{pmatrix}$$

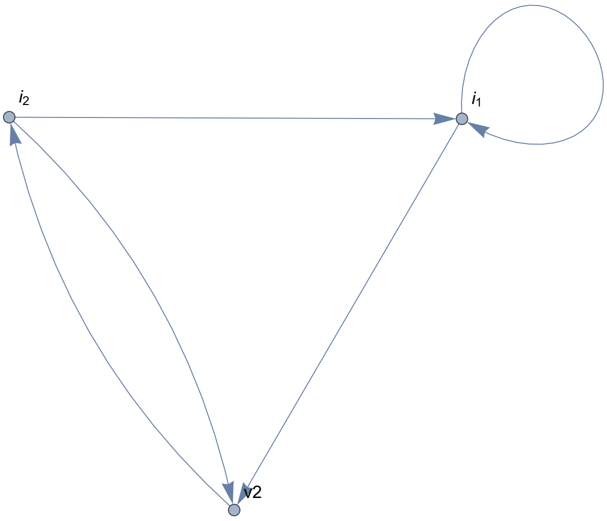
Hermitian adjacency spectrum:

$\{-\sqrt{3}, \sqrt{3}, 1\}$

{-1.73205, 1.73205, 1.}

---

Graph $D_4^{(I)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 0 & -i & 1 \\ i & 1 & -i \\ 1 & i & 0 \end{pmatrix}$$

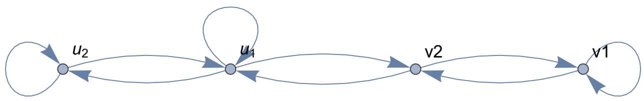
Hermitian adjacency spectrum:

$\{-\sqrt{3}, \sqrt{3}, 1\}$

{-1.73205, 1.73205, 1.}

---

Graph $D_4^{(C)}$




Hermitian adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Hermitian adjacency spectrum:

{Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 4], Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 3],
Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 1], Root[-1 + 4 #1 - 3 #1^3 + #1^4 &, 2]}

{2.35567, 1.47726, -1.09529, 0.26236}

## Skew adjacency spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Skew adjacency spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4"];
skewAdjacencySpecS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(R)"];
skewAdjacencySpecS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(O)"];
skewAdjacencySpecS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(I)"];
skewAdjacencySpecS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(C)"];
skewAdjacencySpecS[d4C]
% // N
```

---------------------------------------------------------------------------

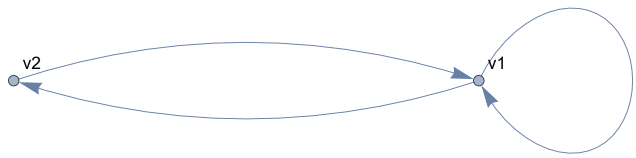
---------------------------------------------------------------------------

Skew adjacency spectra

---------------------------------------------------------------------------

---------------------------------------------------------------------------

Graph $D_4$




Adjacency matrix:

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Skew adjacency matrix:

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

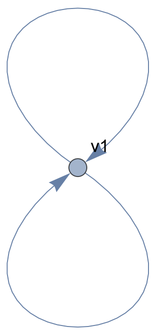
Skew adjacency spectrum:

{0, 0}

{0., 0.}

---------------------------------------------------------------------------

Graph $D_4^{(R)}$




Adjacency matrix:

( 2 )

Skew adjacency matrix:

( 0 )

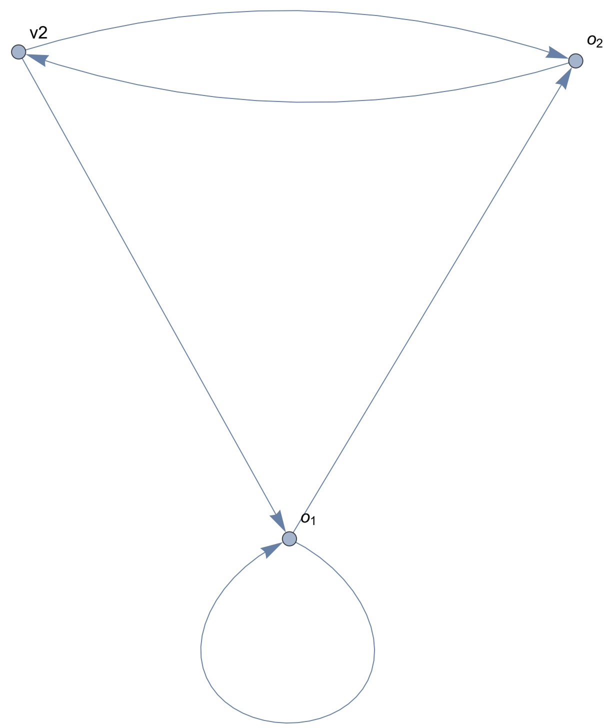
Skew adjacency spectrum:

{0}

{0.}

-----------------------------------------------------------------------

Graph $D_4^{(0)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

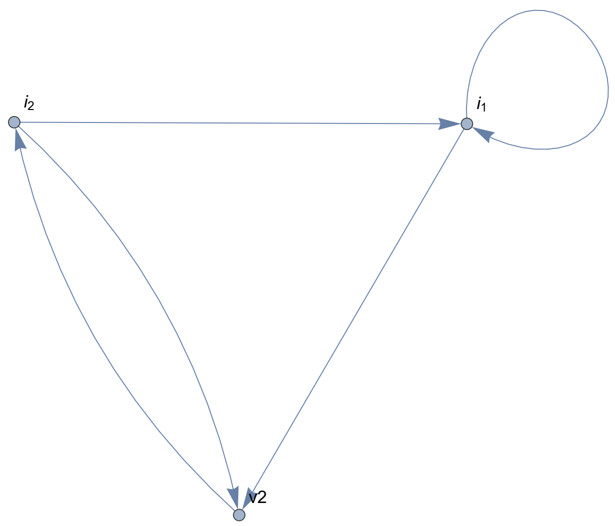
Skew adjacency spectrum:

$\{i\sqrt{2}, -i\sqrt{2}, 0\}$

$\{0. + 1.41421\, i, 0. - 1.41421\, i, 0.\}$

------------------------------------------------------------------------

Graph $D_4^{(I)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

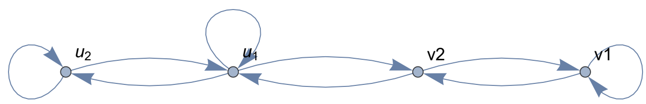
Skew adjacency spectrum:

$\{i\sqrt{2}, -i\sqrt{2}, 0\}$

$\{0. + 1.41421\, i, 0. - 1.41421\, i, 0.\}$

------------------------------------------------------------------------

Graph $D_4^{(C)}$




Adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Skew adjacency spectrum:

$\{0, 0, 0, 0\}$

```
{0., 0., 0., 0.}
```

## Binary skew adjacency spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Binary skew adjacency spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4"];
skewAdjacencySpecBinaryS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4(R)"];
skewAdjacencySpecBinaryS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4(O)"];
skewAdjacencySpecBinaryS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4(I)"];
skewAdjacencySpecBinaryS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4(C)"];
skewAdjacencySpecBinaryS[d4C]
% // N
```

-------------------------------------------------------------------------------

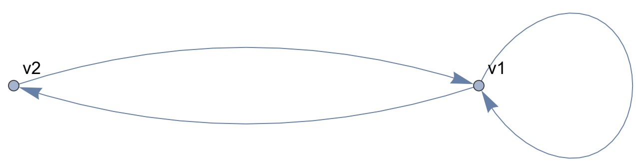
-------------------------------------------------------------------------------

Binary skew adjacency spectra

-------------------------------------------------------------------------------

-------------------------------------------------------------------------------

Graph $D_4$




Binary adjacency matrix:

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Binary skew adjacency matrix:

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

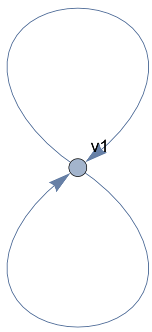
Binary skew adjacency spectrum:

{0, 0}

{0., 0.}

---------------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary adjacency matrix:

( 1 )

Binary skew adjacency matrix:

( 0 )

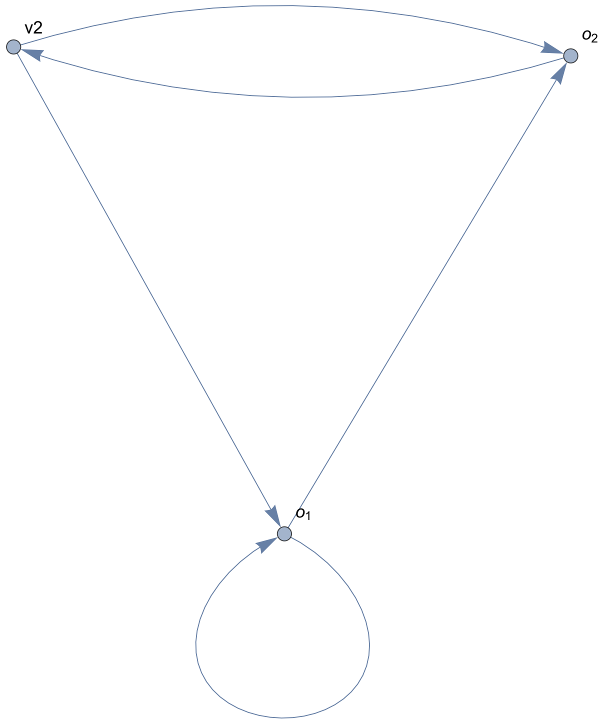
Binary skew adjacency spectrum:

{0}

{0.}

----------------------------------------------------------------------

Graph $D_4^{(0)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

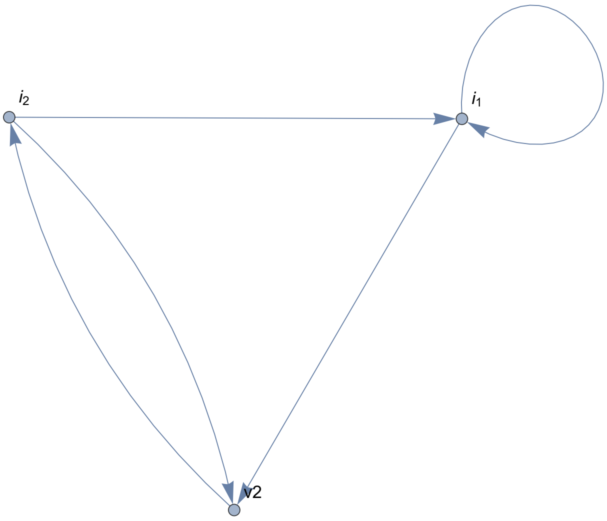
Binary skew adjacency spectrum:

$\{i\sqrt{2}, -i\sqrt{2}, 0\}$

$\{0. + 1.41421\, i, 0. - 1.41421\, i, 0.\}$

------------------------------------------------------------------------

Graph $D_4^{(I)}$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

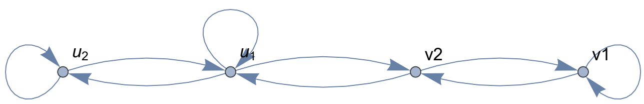
Binary skew adjacency spectrum:

$\{i\sqrt{2}, -i\sqrt{2}, 0\}$

$\{0. + 1.41421\, i, 0. - 1.41421\, i, 0.\}$

------------------------------------------------------------------------

Graph $D_4^{(C)}$




Binary adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency spectrum:

$\{0, 0, 0, 0\}$

```
{0., 0., 0., 0.}
```

## Skew Laplace spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Skew Laplace spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4"];
skewLaplaceSpecS[d4]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(R)"];
skewLaplaceSpecS[d4R]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(O)"];
skewLaplaceSpecS[d4O]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(I)"];
skewLaplaceSpecS[d4I]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(C)"];
skewLaplaceSpecS[d4C]
% // N
```

----------------------------------------------------------------------

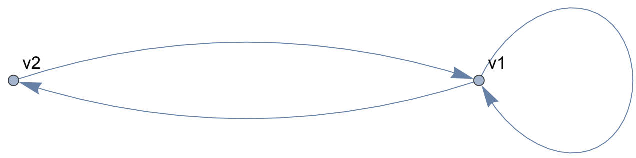
----------------------------------------------------------------------

Skew Laplace spectra

----------------------------------------------------------------------

----------------------------------------------------------------------

Graph $D_4$




Skew Laplacian matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

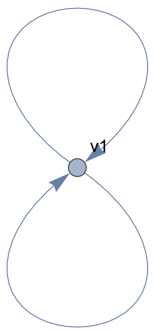
Skew Laplace spectrum

{0, 0}

{0., 0.}

-------------------------------------------------------------------------------

Graph $D_4^{(R)}$




Skew Laplacian matrix

$\begin{pmatrix} 0 \end{pmatrix}$

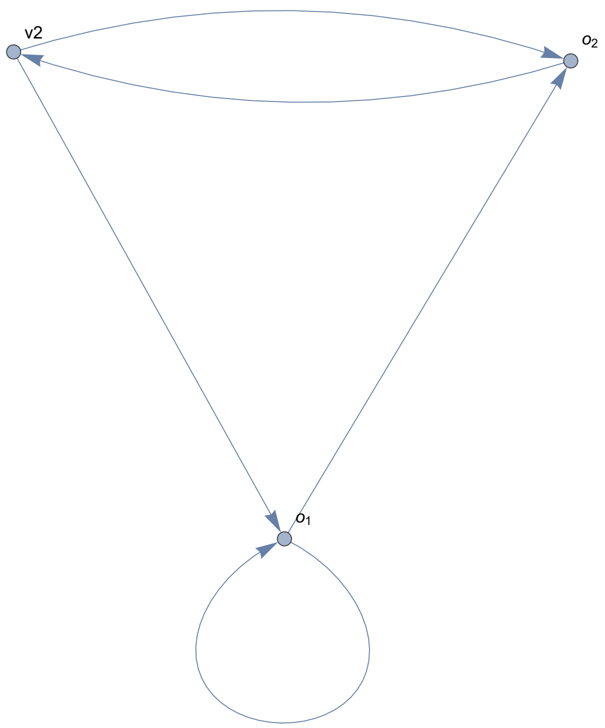
Skew Laplace spectrum

{0}

{0.}

--------------------------------------------------------------------------

Graph $D_4^{(0)}$




Skew Laplacian matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

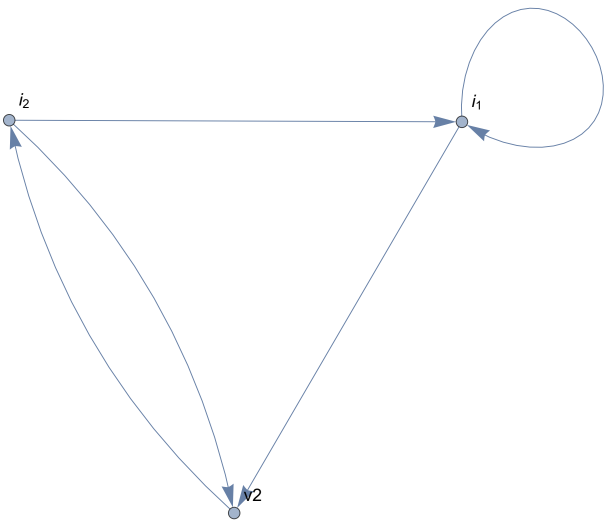
Skew Laplace spectrum

{i, -i, 0}

{0. + 1. i, 0. - 1. i, 0.}

-----------------------------------------------------------------------

Graph $D_4^{(I)}$




Skew Laplacian matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

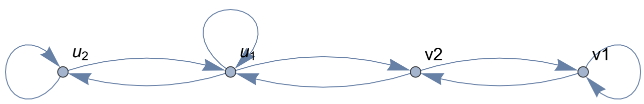
Skew Laplace spectrum

{i, -i, 0}

{0. + 1. i, 0. - 1. i, 0.}

-----------------------------------------------------------------------

Graph $D_4^{(C)}$




Skew Laplacian matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Skew Laplace spectrum

{0, 0, 0, 0}

{0., 0., 0., 0.}

## Binary skew Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Binary skew Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4"];
skewLaplaceSpecBinaryS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
skewLaplaceSpecBinaryS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
skewLaplaceSpecBinaryS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
skewLaplaceSpecBinaryS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
skewLaplaceSpecBinaryS[d4C]
% // N
```

------------------------------------------------------------------------------

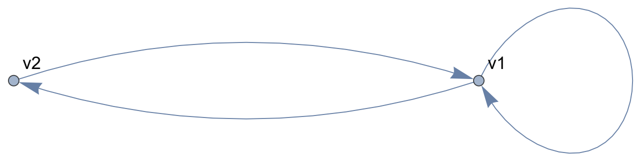
------------------------------------------------------------------------------

Binary skew Laplace spectra

------------------------------------------------------------------------------

------------------------------------------------------------------------------

Graph $D_4$




Binary skew Laplacian matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

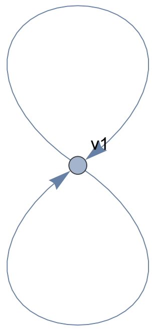
Binary skew Laplace spectrum

{0, 0}

{0., 0.}

------------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary skew Laplacian matrix

$\begin{pmatrix} 0 \end{pmatrix}$

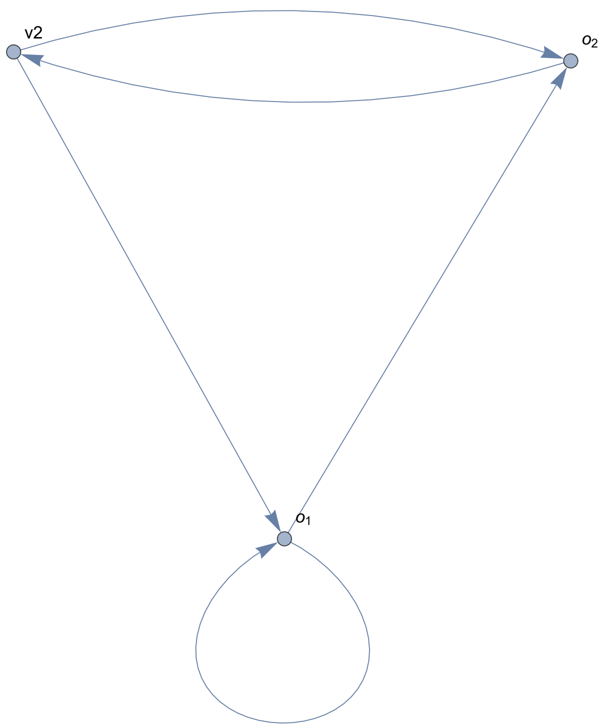
Binary skew Laplace spectrum

{0}

{0.}

-----------------------------------------------------------------------

Graph $D_4^{(0)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

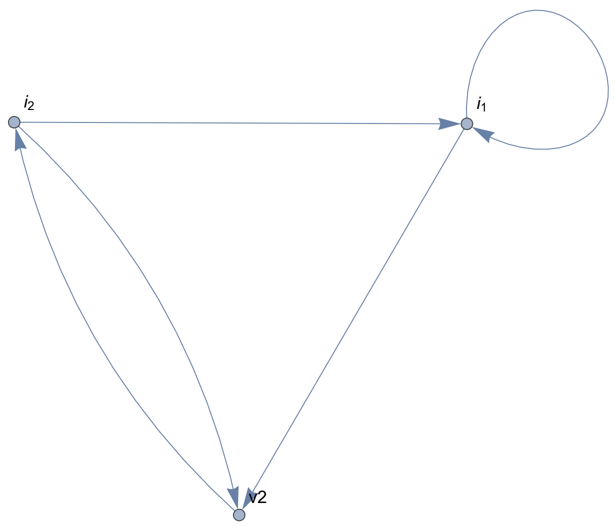
Binary skew Laplace spectrum

{i, -i, 0}

{0. + 1. i, 0. - 1. i, 0.}

------------------------------------------------------------------------

Graph $D_4^{(I)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

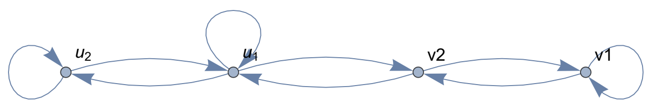
Binary skew Laplace spectrum

{i, -i, 0}

{0. + 1. i, 0. - 1. i, 0.}

------------------------------------------------------------------------

Graph $D_4^{(C)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Binary skew Laplace spectrum

{0, 0, 0, 0}

{0., 0., 0., 0.}

## Normalized Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Normalized Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4"];
normalizedLaplaceSpecS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(R)"];
normalizedLaplaceSpecS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(O)"];
normalizedLaplaceSpecS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(I)"];
normalizedLaplaceSpecS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4^(C)"];
normalizedLaplaceSpecS[d4C]
% // N
```

----------------------------------------------------------------------

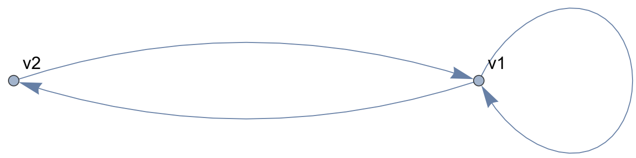
----------------------------------------------------------------------

Normalized Laplace spectra

----------------------------------------------------------------------

----------------------------------------------------------------------

Graph $D_4$




Transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{2}{3}, \frac{1}{3}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

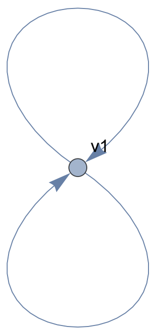
Normalized Laplace spectrum:

$$\left\{\frac{3}{2}, 0\right\}$$

{1.5, 0.}

------------------------------------------------------------------------------

Graph $D_4^{(R)}$




Transition probability matrix:

$\begin{pmatrix} 1 \end{pmatrix}$

Perron–Frobenius vector:

{1}

Perron–Frobenius vector as a matrix:

$\begin{pmatrix} 1 \end{pmatrix}$

Normalized Laplacian:

$\begin{pmatrix} 0 \end{pmatrix}$

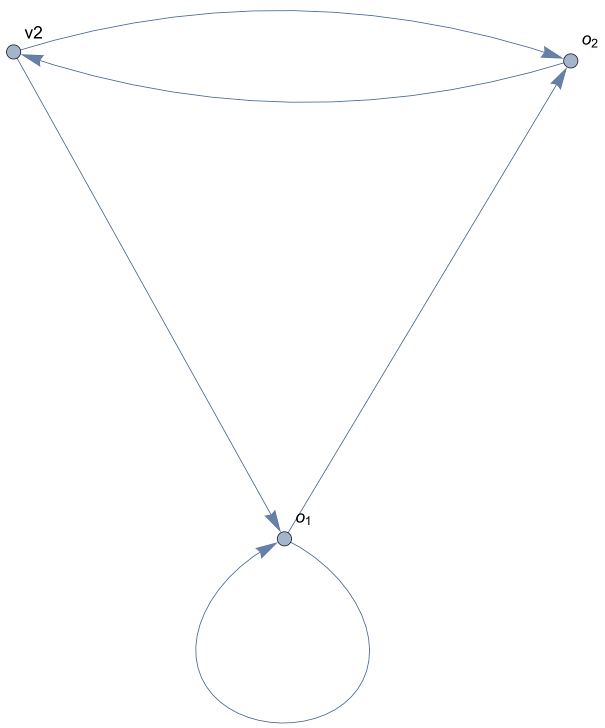
Normalized Laplace spectrum:

{0}

{0.}

----------------------------------------------------------------------

Graph $D_4^{(0)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

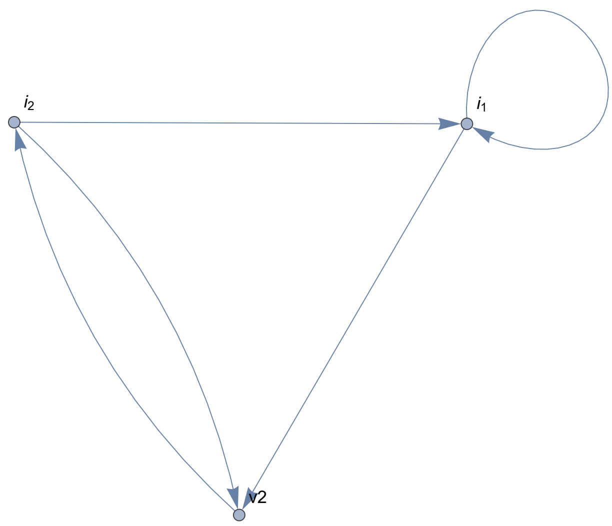
Normalized Laplace spectrum:

$$\left\{\frac{7}{4}, \frac{3}{4}, 0\right\}$$

{1.75, 0.75, 0.}

-----------------------------------------------------------------------

Graph $D_4^{(I)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

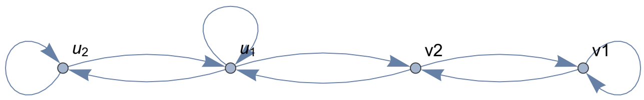
Normalized Laplace spectrum:

$$\left\{\frac{7}{4}, \frac{3}{4}, 0\right\}$$

{1.75, 0.75, 0.}

---

Graph $D_4^{(C)}$




Transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Perron−Frobenius vector:

$$\left\{\frac{2}{9}, \frac{1}{3}, \frac{2}{9}, \frac{2}{9}\right\}$$

Perron−Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{9} & 0 \\ 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

Normalized Laplacian:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{2}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{\sqrt{6}} & 0 & 1 \end{pmatrix}$$

Normalized Laplace spectrum:

$$\left\{\frac{3}{2}, \frac{1}{12}\left(7+\sqrt{13}\right), \frac{1}{12}\left(7-\sqrt{13}\right), 0\right\}$$

{1.5, 0.883796, 0.282871, 0.}

# Combinatorial Laplace spectra

```
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Combinatorial Laplace spectra"];
Print[
  "-------------------------------------------------------------------------------"];
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_4"];
combinatorialLaplaceSpecS[d4]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
combinatorialLaplaceSpecS[d4R]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
combinatorialLaplaceSpecS[d4O]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
combinatorialLaplaceSpecS[d4I]
% // N
Print[
  "-------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
combinatorialLaplaceSpecS[d4C]
% // N
```

-----------------------------------------------------------------------

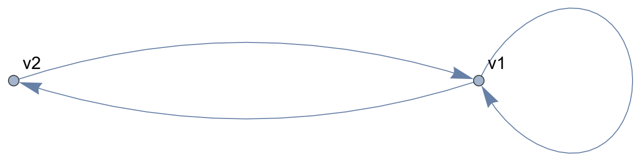
-----------------------------------------------------------------------

Combinatorial Laplace spectra

-----------------------------------------------------------------------

-----------------------------------------------------------------------

Graph $D_4$




Transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{2}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

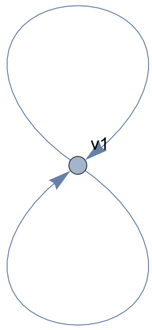
Combinatorial Laplace spectrum:

$$\left\{\frac{2}{3}, 0\right\}$$

{0.666667, 0.}

---------------------------------------------------------------------------

Graph $D_4^{(R)}$




Transition probability matrix:

( 1 )

Perron-Frobenius vector:

{1}

Perron-Frobenius vector as a matrix:

( 1 )

Combinatorial Laplacian:

( 0 )

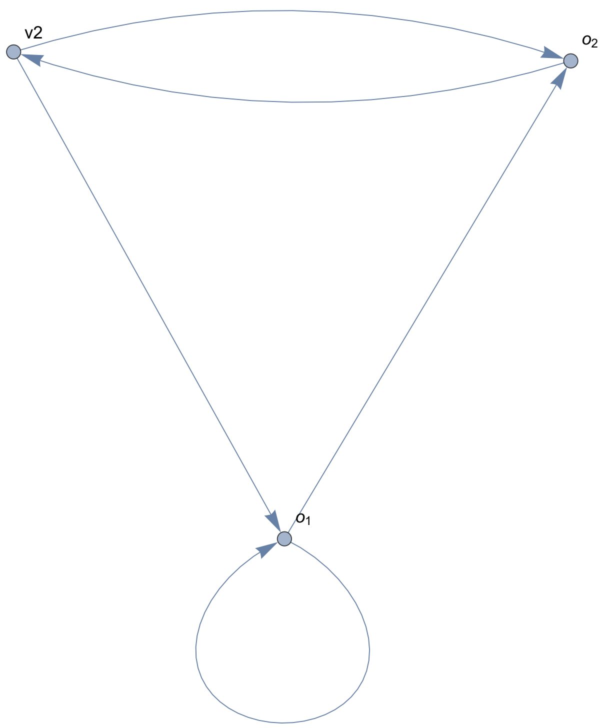
Combinatorial Laplace spectrum:

{0}

{0.}

---------------------------------------------------------------------------

Graph $D_4^{(0)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{12} & -\frac{1}{4} \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{4} & -\frac{1}{12} & \frac{1}{3} \end{pmatrix}$$

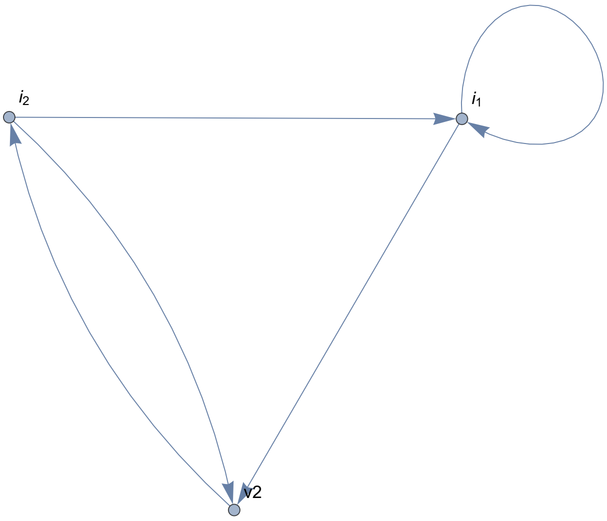
Combinatorial Laplace spectrum:

$$\left\{\frac{7}{12}, \frac{1}{4}, 0\right\}$$

{0.583333, 0.25, 0.}

-----------------------------------------------------------------------

Graph $D_4^{(I)}$




Transition probability matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{12} & -\frac{1}{4} \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{4} & -\frac{1}{12} & \frac{1}{3} \end{pmatrix}$$

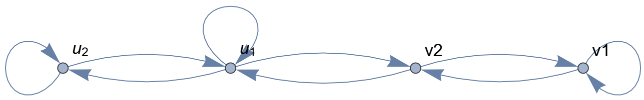
Combinatorial Laplace spectrum:

$$\left\{\frac{7}{12}, \frac{1}{4}, 0\right\}$$

{0.583333, 0.25, 0.}

--------------------------------------------------------------------

Graph $D_4^{(C)}$




Transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{2}{9}, \frac{1}{3}, \frac{2}{9}, \frac{2}{9}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{9} & 0 \\ 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

Combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{9} & 0 & 0 & -\frac{1}{9} \\ 0 & \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & -\frac{1}{9} & \frac{1}{9} & 0 \\ -\frac{1}{9} & -\frac{1}{9} & 0 & \frac{2}{9} \end{pmatrix}$$

Combinatorial Laplace spectrum:

$$\left\{\frac{1}{9}\left(2+\sqrt{2}\right), \frac{2}{9}, \frac{1}{9}\left(2-\sqrt{2}\right), 0\right\}$$

{0.379357, 0.222222, 0.0650874, 0.}

## Binary normalized Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Binary Normalized Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4"];
normalizedLaplaceSpecBinaryS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
normalizedLaplaceSpecBinaryS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
normalizedLaplaceSpecBinaryS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
normalizedLaplaceSpecBinaryS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
normalizedLaplaceSpecBinaryS[d4C]
% // N
```

------------------------------------------------------------------------------

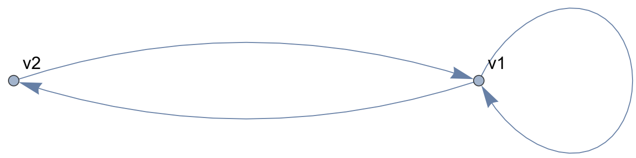
------------------------------------------------------------------------------

Binary Normalized Laplace spectra

------------------------------------------------------------------------------

------------------------------------------------------------------------------

Graph $D_4$




Binary transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{2}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

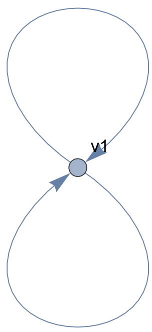
Binary normalized Laplace spectrum:

$$\left\{\frac{3}{2}, 0\right\}$$

{1.5, 0.}

------------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary transition probability matrix:

( 1 )

Perron−Frobenius vector:

{1}

Perron−Frobenius vector as a matrix:

( 1 )

Binary normalized Laplacian:

( 0 )

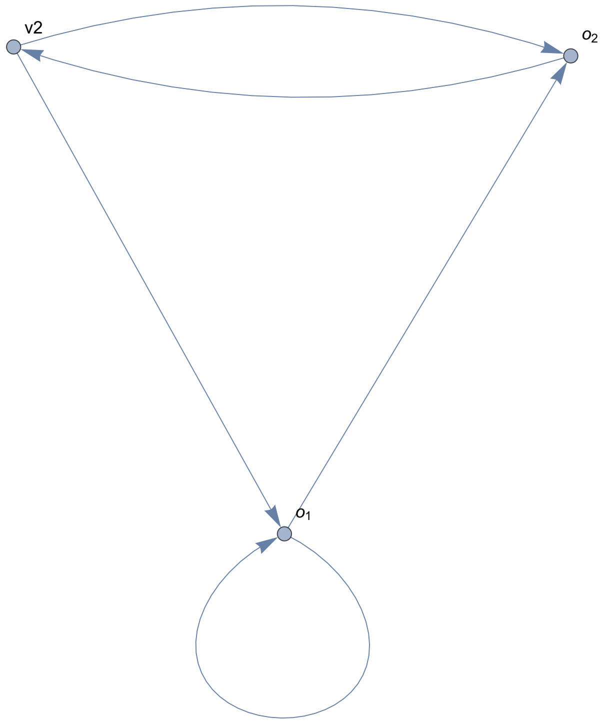
Binary normalized Laplace spectrum:

{0}

{0.}

------------------------------------------------------------------------

Graph $D_4^{(0)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

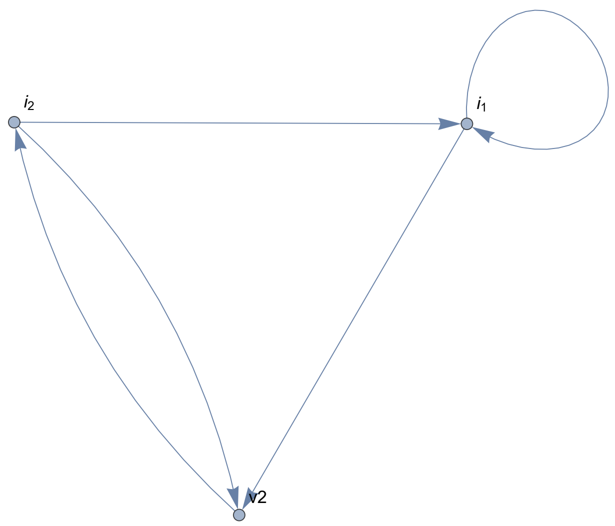
Binary normalized Laplace spectrum:

$$\left\{\frac{7}{4}, \frac{3}{4}, 0\right\}$$

{1.75, 0.75, 0.}

-------------------------------------------------------------------------

Graph $D_4^{(I)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

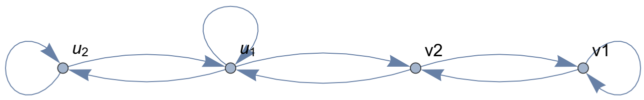
Binary normalized Laplace spectrum:

$$\left\{\frac{7}{4}, \frac{3}{4}, 0\right\}$$

{1.75, 0.75, 0.}

---------------------------------------------------------------------------

Graph $D_4^{(C)}$




Binary transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{2}{9}, \frac{1}{3}, \frac{2}{9}, \frac{2}{9}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{9} & 0 \\ 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

Binary normalized Laplacian:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{2}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{\sqrt{6}} & 0 & 1 \end{pmatrix}$$

Binary normalized Laplace spectrum:

$$\left\{\frac{3}{2}, \frac{1}{12}\left(7+\sqrt{13}\right), \frac{1}{12}\left(7-\sqrt{13}\right), 0\right\}$$

{1.5, 0.883796, 0.282871, 0.}

## Binary combinatorial Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Binary Combinatorial Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4"];
combinatorialLaplaceSpecBinaryS[d4]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(R)"];
combinatorialLaplaceSpecBinaryS[d4R]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(O)"];
combinatorialLaplaceSpecBinaryS[d4O]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(I)"];
combinatorialLaplaceSpecBinaryS[d4I]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D_4^(C)"];
combinatorialLaplaceSpecBinaryS[d4C]
% // N
```

---------------------------------------------------------------------

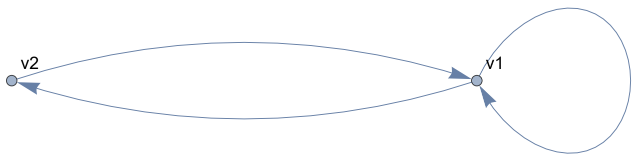
---------------------------------------------------------------------

Binary Combinatorial Laplace spectra

---------------------------------------------------------------------

---------------------------------------------------------------------

Graph $D_4$




Binary transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{2}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

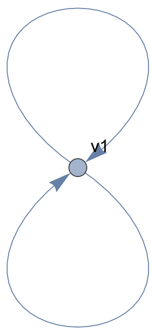
Binary combinatorial Laplace spectrum:

$$\left\{\frac{2}{3}, 0\right\}$$

{0.666667, 0.}

------------------------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary transition probability matrix:

( 1 )

Perron−Frobenius vector:

{1}

Perron−Frobenius vector as a matrix:

( 1 )

Binary combinatorial Laplacian:

( 0 )

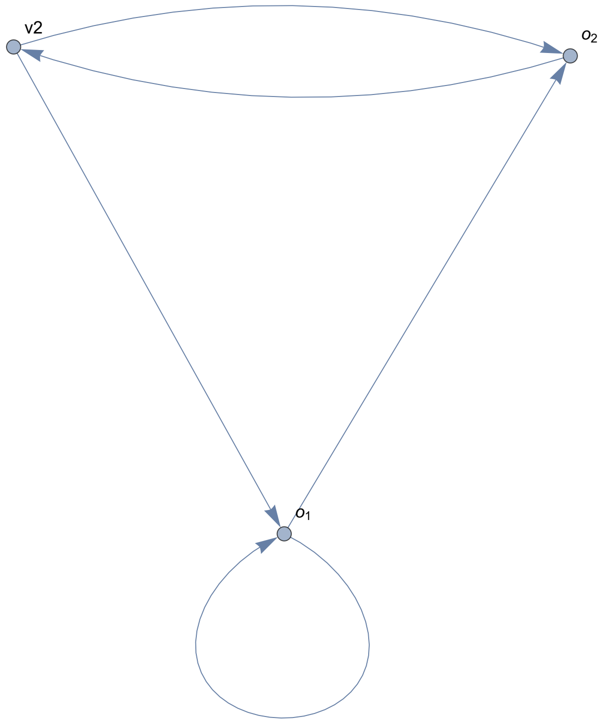
Binary combinatorial Laplace spectrum:

{0}

{0.}

---------------------------------------------------------------------

Graph $D_4^{(O)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

Perron–Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron–Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{12} & -\frac{1}{4} \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{4} & -\frac{1}{12} & \frac{1}{3} \end{pmatrix}$$

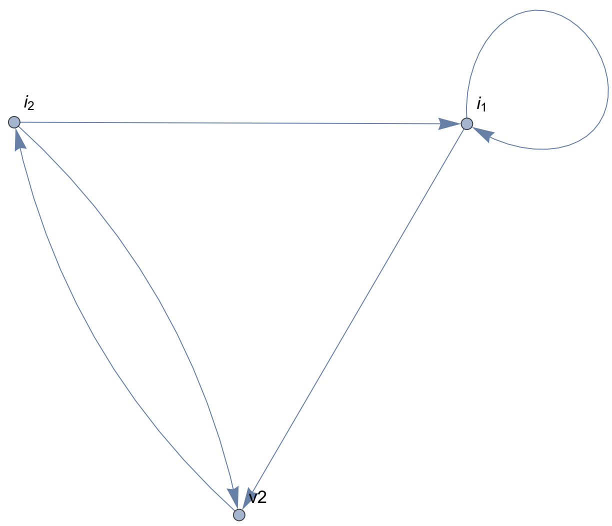
Binary combinatorial Laplace spectrum:

$$\left\{\frac{7}{12}, \frac{1}{4}, 0\right\}$$

{0.583333, 0.25, 0.}

-----------------------------------------------------------------------------

Graph $D_4^{(I)}$




Binary transition probability matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{12} & -\frac{1}{4} \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{4} & -\frac{1}{12} & \frac{1}{3} \end{pmatrix}$$

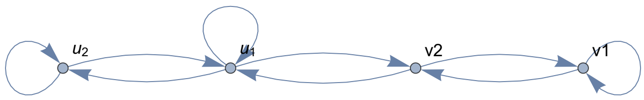
Binary combinatorial Laplace spectrum:

$$\left\{\frac{7}{12}, \frac{1}{4}, 0\right\}$$

{0.583333, 0.25, 0.}

-----------------------------------------------------------------------

Graph $D_4^{(C)}$




Binary transition probability matrix:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Perron-Frobenius vector:

$$\left\{\frac{2}{9}, \frac{1}{3}, \frac{2}{9}, \frac{2}{9}\right\}$$

Perron-Frobenius vector as a matrix:

$$\begin{pmatrix} \frac{2}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{9} & 0 \\ 0 & 0 & 0 & \frac{2}{9} \end{pmatrix}$$

Binary combinatorial Laplacian:

$$\begin{pmatrix} \frac{1}{9} & 0 & 0 & -\frac{1}{9} \\ 0 & \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & -\frac{1}{9} & \frac{1}{9} & 0 \\ -\frac{1}{9} & -\frac{1}{9} & 0 & \frac{2}{9} \end{pmatrix}$$

Binary combinatorial Laplace spectrum:

$$\left\{\frac{1}{9}\left(2+\sqrt{2}\right), \frac{2}{9}, \frac{1}{9}\left(2-\sqrt{2}\right), 0\right\}$$

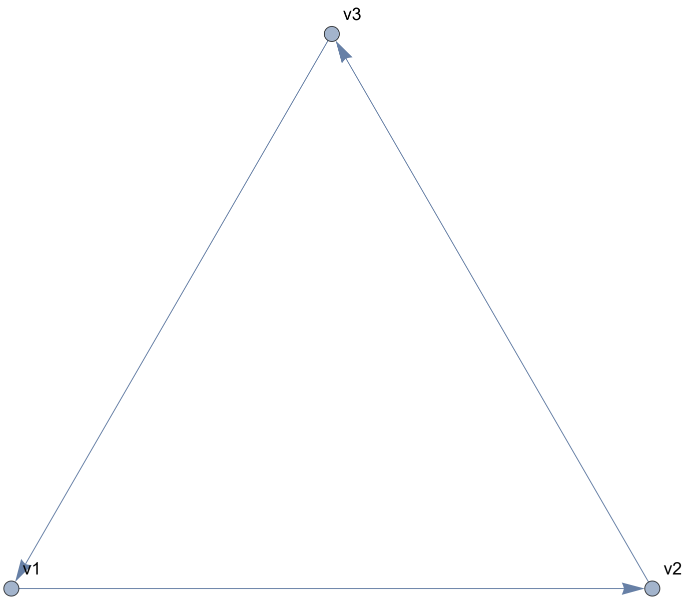
{0.379357, 0.222222, 0.0650874, 0.}

# Example 5.5 (of arXiv:2010.10769 [math.CO] v1)

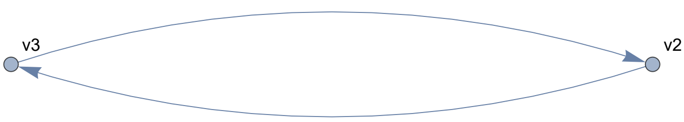
## Definitions of the graphs

```
(* Define the graph D_5 *)
Print["Graph D_5"];
d5 = Graph[{v1, v2, v3}, {v1 ⟷ v2, v2 ⟷ v3, v3 ⟷ v1}, VertexLabels → "Name"]

(* Define the graph D_5^(R) *)
(* computed using d5R=moveR[d5] *)
Print["Graph D_5^(R)"];
d5R = 
```


Graph $D_5$




Graph $D_5^{(R)}$



## Skew adjacency spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Skew adjacency spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D5"];
skewAdjacencySpecS[d5]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4(R)"];
skewAdjacencySpecS[d5R]
% // N
```

---------------------------------------------------------------

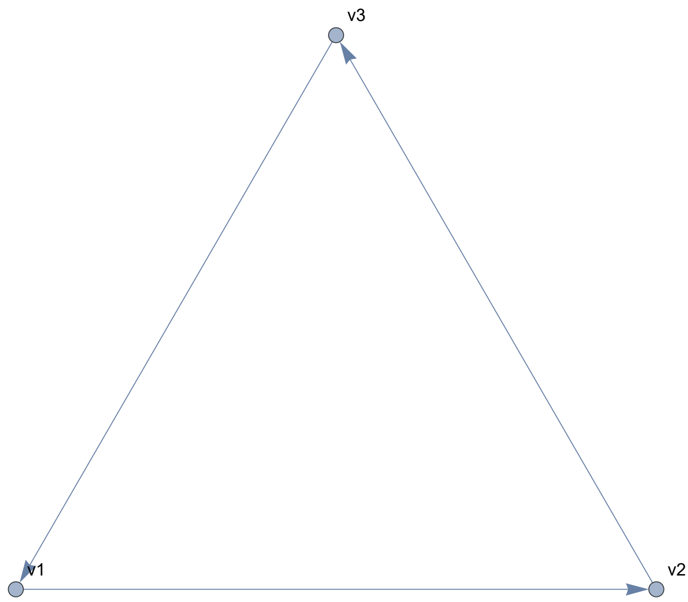
---------------------------------------------------------------

Skew adjacency spectra

---------------------------------------------------------------

---------------------------------------------------------------

Graph $D_5$




Adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

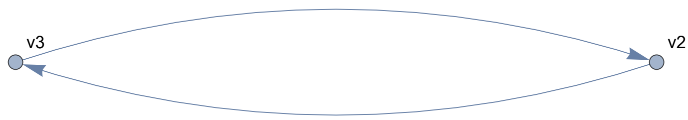
Skew adjacency spectrum:

$\{i\sqrt{3}, -i\sqrt{3}, 0\}$

$\{0. + 1.73205\, i, 0. - 1.73205\, i, 0.\}$

```
-----------------------------------------------------------------------------
```

Graph $D_4^{(R)}$




Adjacency matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Skew adjacency matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Skew adjacency spectrum:

{0, 0}

{0., 0.}

## Binary skew adjacency spectra

```
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Binary skew adjacency spectra"];
Print[
  "-----------------------------------------------------------------------------"];
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D5"];
skewAdjacencySpecBinaryS[d5]
% // N
Print[
  "-----------------------------------------------------------------------------"];
Print["Graph D4(R)"];
skewAdjacencySpecBinaryS[d5R]
% // N
```

------------------------------------------------------------------------------

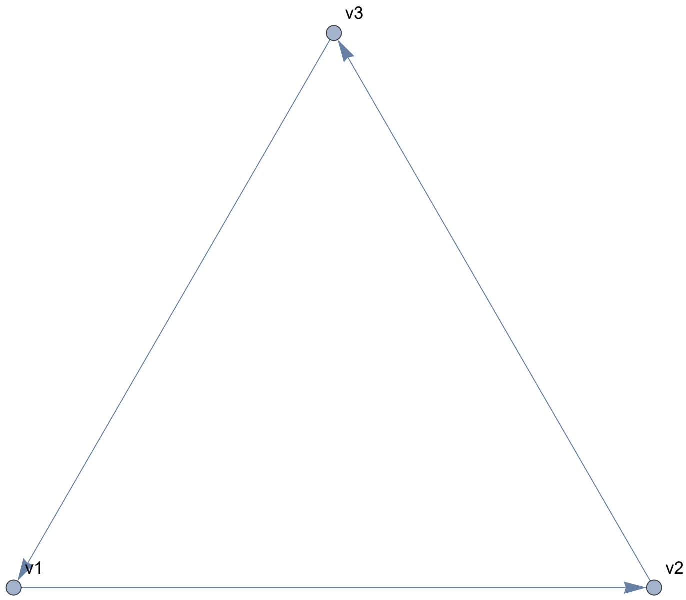
------------------------------------------------------------------------------

Binary skew adjacency spectra

------------------------------------------------------------------------------

------------------------------------------------------------------------------

Graph $D_5$




Binary adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Binary skew adjacency matrix:

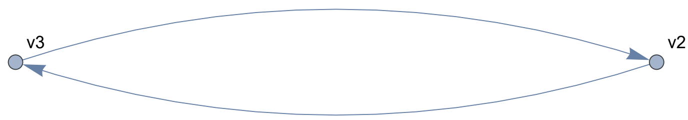
$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Binary skew adjacency spectrum:

$\{i\sqrt{3}, -i\sqrt{3}, 0\}$

$\{0. + 1.73205\, i, 0. - 1.73205\, i, 0.\}$

```
---------------------------------------------------------------------------
Graph D4(R)
```




```
Binary adjacency matrix:
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

```
Binary skew adjacency matrix:
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
Binary skew adjacency spectrum:
{0, 0}
{0., 0.}
```

## Skew Laplace spectra

```
Print[
  "---------------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------------"];
Print["Skew Laplace spectra"];
Print[
  "---------------------------------------------------------------------------"];
Print[
  "---------------------------------------------------------------------------"];
Print["Graph D5"];
skewLaplaceSpecS[d5]
% // N
Print[
  "---------------------------------------------------------------------------"];
Print["Graph D4(R)"];
skewLaplaceSpecS[d5R]
% // N
```

-----------------------------------------------------------------------

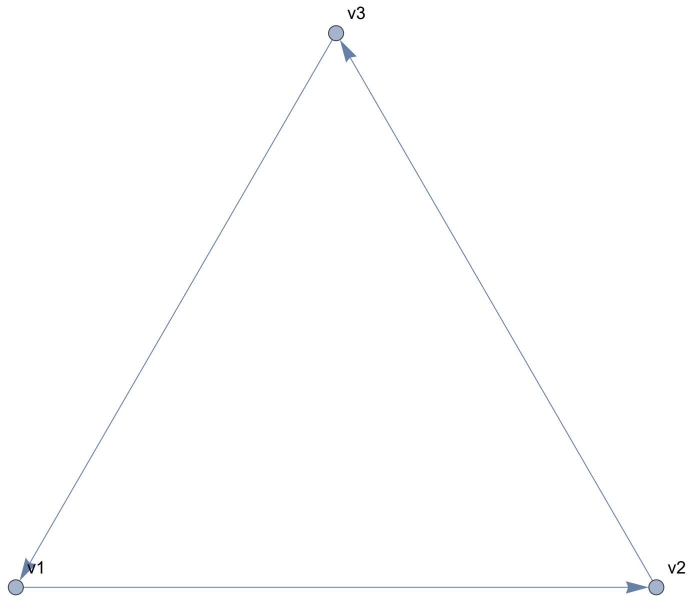
-----------------------------------------------------------------------

Skew Laplace spectra

-----------------------------------------------------------------------

-----------------------------------------------------------------------

Graph $D_5$




Skew Laplacian matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

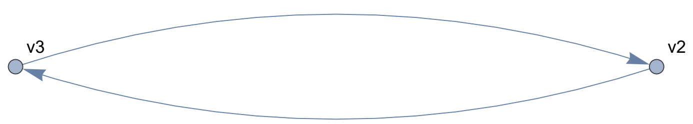
Skew Laplace spectrum

$\{i\sqrt{3}, -i\sqrt{3}, 0\}$

$\{0. + 1.73205\, i, 0. - 1.73205\, i, 0.\}$

-----------------------------------------------------------------------

Graph $D_4^{(R)}$




Skew Laplacian matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Skew Laplace spectrum

$\{0, 0\}$

$\{0., 0.\}$

## Binary skew Laplace spectra

```
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Binary skew Laplace spectra"];
Print[
  "------------------------------------------------------------------------------"];
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D5"];
skewLaplaceSpecBinaryS[d5]
% // N
Print[
  "------------------------------------------------------------------------------"];
Print["Graph D4(R)"];
skewLaplaceSpecBinaryS[d5R]
% // N
```

------------------------------------------------------------------------

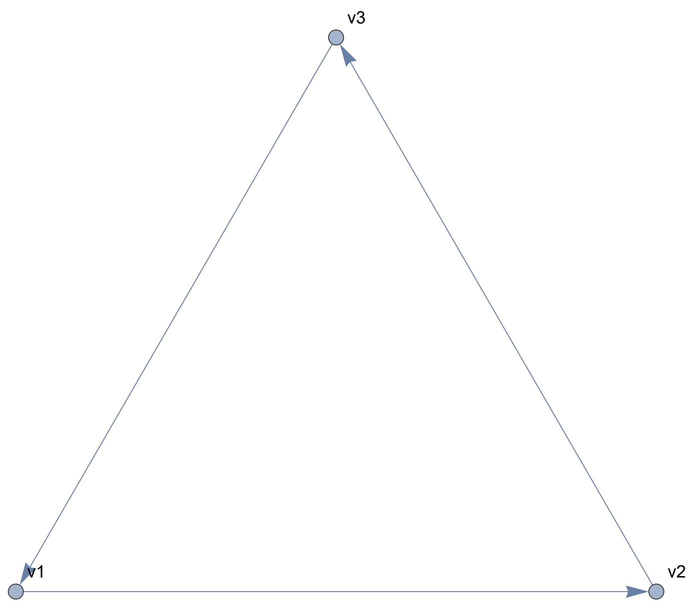
------------------------------------------------------------------------

Binary skew Laplace spectra

------------------------------------------------------------------------

------------------------------------------------------------------------

Graph $D_5$




Binary skew Laplacian matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

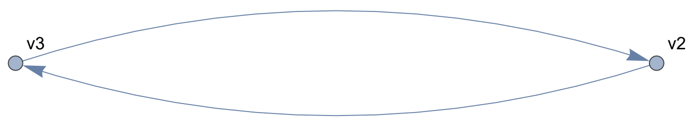
Binary skew Laplace spectrum

$\{i\sqrt{3}, -i\sqrt{3}, 0\}$

{0. + 1.73205 i, 0. - 1.73205 i, 0.}

-----------------------------------------------------------------------------

Graph $D_4^{(R)}$




Binary skew Laplacian matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Binary skew Laplace spectrum

{0, 0}

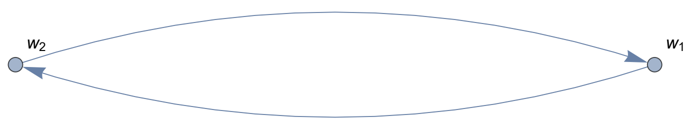
{0., 0.}

# Example 5.6 (of arXiv:2010.10769 [math.CO] v1)

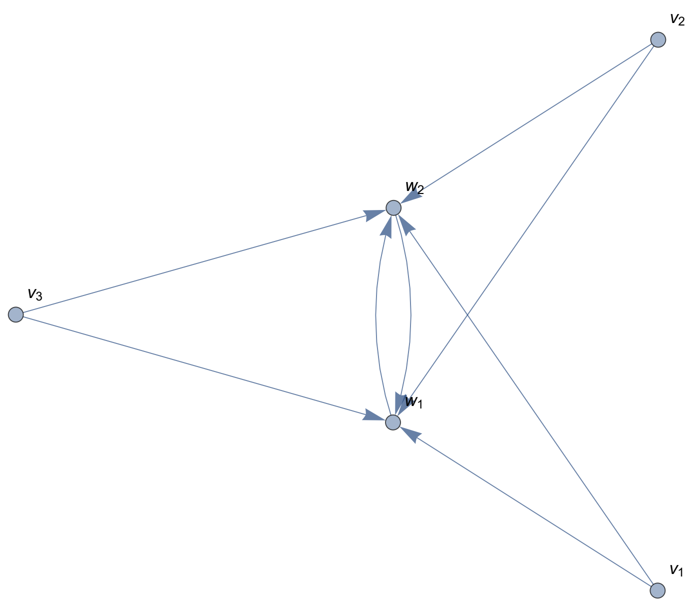
## Definitions of the graphs (m = 3)

```
m = 3;

(* Define the graph D0 *)
Print["Graph D0"];
d0 =
 Graph[{w1, w2}, {DirectedEdge[w1, w2], DirectedEdge[w2, w1]}, VertexLabels -> "Name"]

(* Define the graph Dm *)
Print["Graph Dm"];
dm = Graph[Join[{w1, w2}, Table[vi, {i, 1, m}]],
  Flatten[Join[{DirectedEdge[w1, w2], DirectedEdge[w2, w1]}, Table[
     {DirectedEdge[vi, w1], DirectedEdge[vi, w2]}, {i, 1, m}]]], VertexLabels -> "Name"]
```

Graph $D_0$



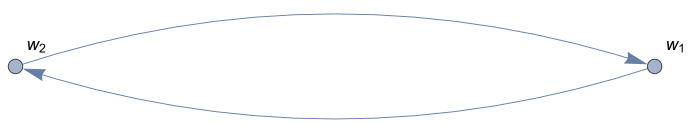

Graph $D_m$



## Laplace spectra (m = 3)

```
Print[
  "----------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------"];
Print["Laplace spectra"];
Print[
  "----------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------"];
Print["Graph D0"];
laplaceSpecS[d0]
Print[
  "----------------------------------------------------------------------------"];
Print["Graph Dm"];
laplaceSpecS[dm]
```

```
----------------------------------------------------------------------------

----------------------------------------------------------------------------

Laplace spectra

----------------------------------------------------------------------------

----------------------------------------------------------------------------

Graph D0
```




Incidence matrix:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Laplacian:

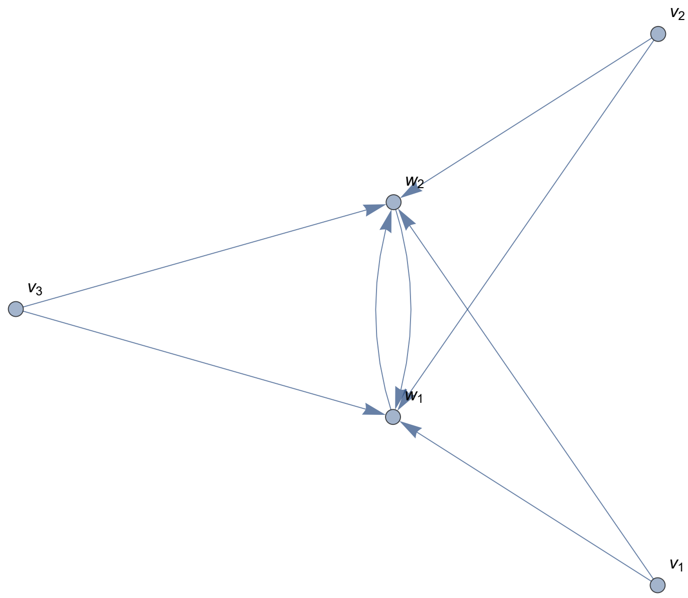
$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Laplace spectrum:

{4, 0}

------------------------------------------------------------------------

Graph $D_m$




Incidence matrix:

$$\begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Laplacian:

$$\begin{pmatrix} 5 & -2 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

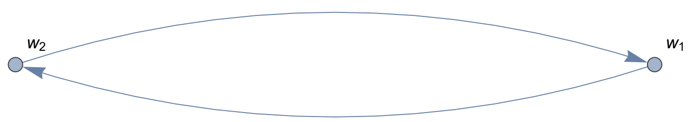
Laplace spectrum:

{7, 5, 2, 2, 0}

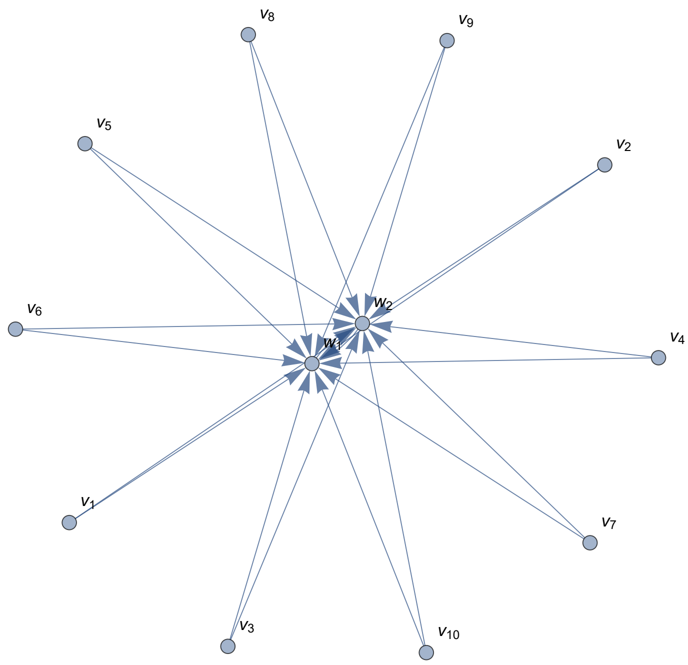
## Definitions of the graphs (m = 10)

```
m = 10;

(* Define the graph D0 *)
Print["Graph D0"];
d0 =
 Graph[{w1, w2}, {DirectedEdge[w1, w2], DirectedEdge[w2, w1]}, VertexLabels -> "Name"]

(* Define the graph Dm *)
Print["Graph Dm"];
dm = Graph[Join[{w1, w2}, Table[vi, {i, 1, m}]],
   Flatten[Join[{DirectedEdge[w1, w2], DirectedEdge[w2, w1]}, Table[
      {DirectedEdge[vi, w1], DirectedEdge[vi, w2]}, {i, 1, m}]]], VertexLabels -> "Name"]
```

Graph $D_0$



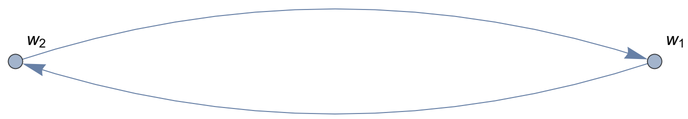

Graph $D_m$



## Laplace spectra (m = 10)

```
Print[
  "----------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------"];
Print["Laplace spectra"];
Print[
  "----------------------------------------------------------------------------"];
Print[
  "----------------------------------------------------------------------------"];
Print["Graph D0"];
laplaceSpecS[d0]
Print[
  "----------------------------------------------------------------------------"];
Print["Graph Dm"];
laplaceSpecS[dm]
```

```
----------------------------------------------------------------------------

----------------------------------------------------------------------------

Laplace spectra

----------------------------------------------------------------------------

----------------------------------------------------------------------------

Graph D0
```




Incidence matrix:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Laplacian:

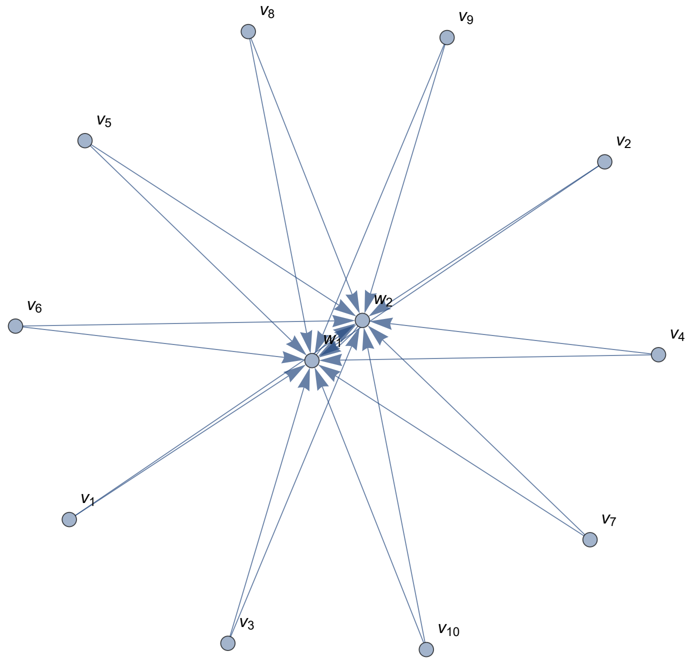
$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Laplace spectrum:

{4, 0}

------------------------------------------------------------------

Graph $D_m$




Incidence matrix:

$$\begin{pmatrix}
1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}$$

Laplacian:

$$\begin{pmatrix}
12 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-2 & 12 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}$$

Laplace spectrum:

{14, 12, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0}