Groupoid actions on graphs and C^*-correspondences (preliminary report)

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Outline

- Let the groupoid G act on a C^*-correspondence $\mathcal{H}$ over the $C_0(G^0)$-algebra A.
- By the universal property G acts on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ which also becomes a $C_0(G^0)$-algebra.
- We study the crossed product $\mathcal{O}_{\mathcal{H}} \rtimes G$ and the fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$.
- Using intertwiners, we define the Doplicher-Roberts algebra $\mathcal{O}_\rho$ of a representation ρ of a groupoid G on $\mathcal{H}$ and prove that under certain conditions $\mathcal{O}_{\mathcal{H}}^G$ is isomorphic to $\mathcal{O}_\rho$.
- Suppose G has finite isotropy groups. If G acts on a discrete locally finite graph E, we prove that the crossed product $C^*(E) \rtimes G$ is stably isomorphic to a locally finite graph algebra.
- We illustrate with some examples.

Groupoid actions

- We assume that G is a locally compact Hausdorff groupoid with unit space G^0 and range and source maps $r, s : G \to G^0$.
- Recall that G acts on a locally compact space X if there are a continuous, open surjection $\pi : X \to G^0$ and a continuous map

$$G * X \to X, \quad \text{write} \quad (g, x) \mapsto g \cdot x,$$

where $G * X = \{(g, x) \in G \times X \mid s(g) = \pi(x)\}$, such that

$$\pi(g \cdot x) = r(g), \;\; g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x, \;\; \pi(x) \cdot x = x.$$

- For example, if G is an étale groupoid, then G acts on its unit space G^0. G also acts on itself by left multiplication.
- One can talk about the set of fixed points X^G, stabilizers $G(x) = \{g \in G \mid g \cdot x = x\}$, orbits $Gx = G * \{x\}$, orbit space X/G etc.
- Note that $G(x)$ is a subgroup of the isotropy group G_u^u, where $u = \pi(x)$.

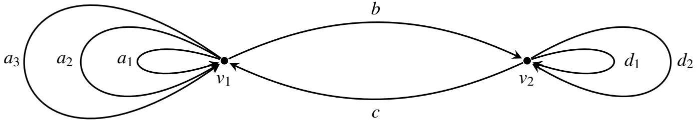
Groupoid actions

- Sometimes a graph has only local symmetries, which motivates the introduction of groupoid actions on graphs.
- Let $E = (E^0, E^1, r, s)$ be a topological graph, i.e. E^0, E^1 are locally compact spaces with $r, s : E^1 \to E^0$ continuous maps and s a local homeomorphism.
- We say that the groupoid G acts on E if G acts on both E^0, E^1 in a compatible way.
- This means that there are maps $\pi^0 : E^0 \to G^0, \pi^1 : E^1 \to G^0$ and maps $G * E^0 \to E^0, G * E^1 \to E^1$ as above such that

$$s(g \cdot e) = g \cdot s(e), r(g \cdot e) = g \cdot r(e) \quad \text{for } e \in E^1 \text{ and } g \in G.$$

Examples

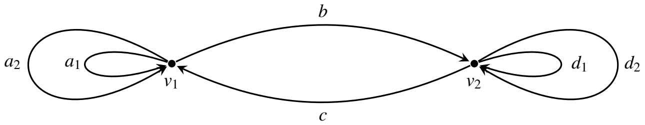
- Let E be the graph with $E^0 = \{v_1, v_2\}$ and $E^1 = \{a_1, a_2, a_3, b, c, d_1, d_2\}$ where $s(a_i) = r(a_i) = s(b) = r(c) = v_1, i = 1, 2, 3, s(d_j) = r(d_j) = r(b) = s(c) = v_2, j = 1, 2.$



- The group bundle $G = S_3 \sqcup S_2$ with $G^0 = \{u_1, u_2\}$ acts on E by permuting the edges a_i for $i = 1, 2, 3$ and d_j for $j = 1, 2$ and fixing b, c. Here $\pi^0 : E^0 \to G^0, \pi^0(v_i) = u_i, i = 1, 2, \pi^1 : E^1 \to G^0, \pi^1 = \pi^0 \circ s.$

Examples

- Let E be the graph with $E^0 = \{v_1, v_2\}$ and $E^1 = \{a_1, a_2, b, c, d_1, d_2\}$ where $s(a_i) = r(a_i) = s(b) = r(c) = v_1, i = 1, 2, s(d_j) = r(d_j) = r(b) = s(c) = v_2, j = 1, 2.$



- Let G be the transitive groupoid with $G^0 = \{u_1, u_2\}$ with isotropy groups $G_{u_i}^{u_i} = \{u_i, t_i\} \cong \mathbb{Z}_2, i = 1, 2$, so $t_i^2 = u_i$. Let $x \in G$ be such that $s(x) = x^{-1}x = u_1, r(x) = xx^{-1} = u_2$ and $xt_1x^{-1} = t_2$.
- Let G act on E such that

$$x \cdot v_1 = v_2, t_1 \cdot a_1 = a_2, t_2 \cdot d_1 = d_2, x \cdot a_i = d_i, i = 1, 2, x \cdot b = c.$$

Here $\pi^0 : E^0 \to G^0, \pi^0(v_i) = u_i$ and $\pi^1 : E^1 \to G^0, \pi^1 = \pi^0 \circ s$.

Groupoid actions on C^*-algebras

- Let X be a locally compact Hausdorff space. Recall that a $C_0(X)-$algebra is a C^*-algebra A together with a homomorphism $\theta : C_0(X) \to ZM(A)$ such that $\theta(C_0(X))A = A$.
- For each $x \in X$ we can define the fiber A_x as A/I_xA where $I_x = \{f \in C_0(X) : f(x) = 0\}$.
- We say that the groupoid G with unit space G^0 acts on a C^*-algebra A if A is a $C_0(G^0)$-algebra and for each $g \in G$ there is an isomorphism $\alpha_g : A_{s(g)} \to A_{r(g)}$ such that if $(g_1, g_2) \in G^{(2)}$ we have $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$.
- We also write $g \cdot a$ for $\alpha_g(a)$.

Groupoid representations on C^*-correspondences

- Let A be a $C_0(X)$-algebra and let $\mathcal{H}$ be a Hilbert A-module. We define the fibers $\mathcal{H}_x := \mathcal{H} \otimes_A A_x$. For $T \in \mathcal{L}(\mathcal{H})$, let $T_x \in \mathcal{L}(\mathcal{H}_x)$ be $T \otimes 1_{A_x}$.
- The set $\mathrm{Iso}(\mathcal{H})$ of $\mathbb{C}$-linear isomorphisms between fibers becomes a groupoid with unit space X.
- If $\mathcal{H}$ is a C^*-correspondence over the $C_0(X)$-algebra A and G is a groupoid with $G^0 = X$ acting on A, a representation of G on $\mathcal{H}$ is given by a groupoid homomorphism $\rho : G \to \mathrm{Iso}(\mathcal{H})$ where $\rho(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ with $\rho(g) = I$ if $g \in G^0$, such that

$$\langle \rho(g)\xi, \rho(g)\eta \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)\xi)(g \cdot a), \;\; \rho(g)(a\xi) = (g \cdot a)(\rho(g)\xi).$$

- If G is a locally compact groupoid with Haar system, then it acts on the $C_0(G^0)$–$C_0(G^0)$ C^*-correspondence with fibers $L^2(G^u, \lambda^u)$ via the left regular representation.
- If the groupoid G acts on a graph E then G acts on the associated $C_0(E^0)$-$C_0(E^0)$ correspondence $\mathcal{H}_E$. Subgroupoids of $\mathrm{Iso}(\mathcal{H}_E)$ (with the same unit space) are candidates to act on $C^*(E)$.

Doplicher-Roberts algebras

- Let the groupoid G act on a A–A C^*-correspondence $\mathcal{H}$ via the representation $\rho : G \to \mathrm{Iso}(\mathcal{H})$.
- Consider the tensor power $\rho^n : G \to \mathrm{Iso}(\mathcal{H}^{\otimes n})$ and let

$$(\rho^m, \rho^n) = \{T : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes m} \mid T_{r(g)}\rho^n(g) = \rho^m(g)T_{s(g)}\}.$$

- The linear span ${}^0\mathcal{O}_\rho$ of $\bigcup\limits_{m,n}(\rho^m, \rho^n)$ has a natural multiplication and involution, after identifying T with $T \otimes I$.
- We assume that

$$\|T\| = \sup\{\|\pi(T)\| : \ \pi \text{ is a } * -\text{rep of } {}^0\mathcal{O}_\rho \text{ on a Hilbert space}\} < \infty.$$

- The Doplicher-Roberts algebra $\mathcal{O}_\rho$ is defined as the C^*-closure of ${}^0\mathcal{O}_\rho$.

Doplicher-Roberts algebras

- **Theorem**. Let $\mathcal{H}$ be a full finite projective C^*-correspondence over A and assume that the left multiplication $A \to \mathcal{L}(\mathcal{H})$ is injective. If G is a groupoid acting on $\mathcal{H}$ via $\rho : G \to \mathrm{Iso}(\mathcal{H})$, then the Doplicher-Roberts algebra $\mathcal{O}_\rho$ is well defined and it is isomorphic to the fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$.
- **Proof**. It is known that $\mathcal{L}(\mathcal{H}) \cong \mathcal{K}(\mathcal{H})$ and that $\mathcal{O}_{\mathcal{H}}$ is isomorphic to the C^*-algebra generated by the span of $\bigcup\limits_{m,n\geq 0} \mathcal{K}(\mathcal{H}^{\otimes m}, \mathcal{H}^{\otimes n})$ after we identify T with $T \otimes I$.
- Note that G acts on $\mathcal{K}(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m})$ by
$$(g \cdot T_{r(g)})(\xi) = \rho^m(g) T_{s(g)}(\rho^n(g^{-1})\xi)$$
and the fixed point algebra is (ρ^m, ρ^n).
- It follows that ${}^0\mathcal{O}_\rho \subseteq \mathcal{O}_{\mathcal{H}}$ and that $\mathcal{O}_\rho$ is isomorphic to $\mathcal{O}_{\mathcal{H}}^G$.

Crossed products of C^*-correspondences

- Let G be a locally compact groupoid with Haar system acting on a $C_0(G^0)$-algebra A. If $\mathcal{H}$ is a A-A C^*-correspondence, then a representation of G on $\mathcal{H}$ determines an action on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$.
- The crossed product $\mathcal{H} \rtimes G = \mathcal{H} \otimes_A (A \rtimes G)$ becomes a C^*-correspondence over $A \rtimes G$ using the operations

$$\langle \xi, \eta \rangle(h) = \int g^{-1} \cdot \langle \xi(g), \eta(gh) \rangle_{r(g)} d\lambda^{r(h)}(g),$$

$$(\xi \cdot f)(h) = \int \xi(g)(g \cdot f(g^{-1}h)) d\lambda^{r(h)}(g),$$

$$(f \cdot \xi)(h) = \int f(g) \cdot (g \cdot \xi(g^{-1}h)) d\lambda^{r(h)}(g),$$

where $\xi \in C_c(G, \mathcal{H})$, $f \in C_c(G, A)$.

Finite isotropy groupoid actions

- **Lemma**. Suppose A and B are SME C^*-algebras with A-B imprimitivity bimodule $\mathcal{X}$.
- If $\mathcal{H}$ is a C^*-correspondence over A, then $\mathcal{H}' = \mathcal{X}^* \otimes_A \mathcal{H} \otimes_A \mathcal{X}$ is a C^*-correspondence over B such that $\mathcal{O}_{\mathcal{H}}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME.
- **Theorem**. Given a locally finite discrete graph E and a groupoid G with finite isotropy groups acting on E, the crossed product $C^*(E) \rtimes G$ is SME to a graph C^*-algebra, where the number of vertices is the cardinality of the spectrum of $C_0(E^0) \rtimes G$.
- **Proof**. Recall that separable non-degenerate C^*-correspondences over $C_0(V)$ with V at most countable determine a discrete graph.
- We use orbits and stabilizers to decompose $\mathcal{H}_E \rtimes G$ over the C^*-algebra $C_0(E^0) \rtimes G$ which is SME to a commutative C^*-algebra.
- It follows that $C^*(E) \rtimes G$ is isomorphic to the C^*-algebra of a graph of (minimal) C^*-correspondences, hence SME to a graph C^*-algebra.

Graphs of C^*-correspondences

- Given a discrete graph $E = (E^0, E^1, r, s)$, associate to each vertex $v \in E^0$ a C^*-algebra A_v and to each edge $e \in E^1$ a nondegenerate $A_{r(e)}$–$A_{s(e)}$ C^*-correspondence $\mathcal{H}_e$.
- This way we obtain an E-system of C^*-correspondences or a graph of C^*-correspondences.
- The C^*-algebra associated to this graph of C^*-correspondences is $\mathcal{O}_{\mathcal{H}}$, where $\mathcal{H} = \bigoplus_{e \in E^1} \mathcal{H}_e$ becomes a C^*-correspondence over $A = \bigoplus_{v \in E^0} A_v$ in a natural way.
- **Example**. Assume we have a C^*-correspondence $\mathcal{H}$ over a unital C^*-algebra A such that A decomposes into a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$.
- If p_j is the identity of A_j, then $\mathcal{H}$ decomposes into a direct sum $\bigoplus_{i,j} p_i \mathcal{H} p_j$ and we can construct a graph of C^*-correspondences with n vertices $\{v_1, ..., v_n\}$, by assigning the C^*-algebra A_i at v_i and the C^*-correspondence $p_i \mathcal{H} p_j \neq 0$ at the edge joining v_j with v_i.

The C^*-correspondence of a groupoid representation

- Let G be a locally compact groupoid with a Haar system λ and let $A = C^*(G)$.
- Given a Hilbert bundle $\mathcal{H}$ over the unit space G^0 and a representation $\rho : G \to \mathrm{Iso}(\mathcal{H})$, denote by $C_c(G, r^*\mathcal{H})$ the space of continuous sections with compact support of the pull-back bundle $r^*\mathcal{H}$.
- We define the left and right actions of $C_c(G)$ on $C_c(G, r^*\mathcal{H})$ by

$$f\xi(g) = \int f(h)\rho(h)\xi(h^{-1}g)d\lambda^{r(g)}(h),$$

$$\xi f(g) = \int \xi(gh)f(h^{-1})d\lambda^{s(g)}(h),$$

  and inner product

$$\langle \xi, \eta \rangle(g) = \int \langle \xi(h^{-1}), \eta(h^{-1}g)\rangle_{s(h)} d\lambda^{r(g)}(h)$$

  for $f \in C_c(G)$ and $\xi, \eta \in C_c(G, r^*\mathcal{H})$.
- The completion $\mathcal{M}$ of $C_c(G, r^*\mathcal{H})$ becomes an A–A correspondence, where the left action of $C_c(G)$ extends to a $*$-homomorphsim $A \to \mathcal{L}_A(\mathcal{M})$. A similar construction could be done using $A = C^*_r(G)$.

The C^*-correspondence of a groupoid representation

- Given representations $\rho_i : G \to \mathrm{Iso}(\mathcal{H}_i)$ for $i = 1, 2$, one may consider the tensor product representation $\rho_1 \otimes \rho_2 : G \to \mathrm{Iso}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and the C^*-correspondence of $\rho_1 \otimes \rho_2$ is the composition of the C^*-correspondences for ρ_1, ρ_2.
- For $G = G^0 = X$ compact and $\mathcal{V}$ a vector bundles over X, this construction provides a $C(X)$–$C(X)$ corespondence $\mathcal{M}$ such that the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{M}}$ is a continuous field of Cuntz algebras.
- For G a compact group and $\rho : G \to U(n)$ a unitary representation, the $C^*(G)$–$C^*(G)$ correspondence $\mathcal{M}$ is isomorphic to $\mathbb{C}^n \otimes C^*(G)$ and $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\mathcal{O}_n \rtimes G$.