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October 17-18, 2015

Orbifold versus Manifold Spectral Theory

Special Session on The Analysis, Geometry, and Topology of Groupoids - GroupoidFest 2015

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joint work with Emily Proctor and Christopher Seaton

Preliminary Report

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Table of Contents

Orbifolds, Isospectrality and Γ-Isospectrality

Orbifolds and Isospectrality

An orbifold $M \rtimes G$ is a topological space that is locally modeled on $\tilde{U} \rtimes K$ where K is a finite subgroup of the orthogonal group $O(n)$ acting on $\tilde{U} \subset \mathbb{R}^n$.

Orbifolds and Isospectrality

General Question:

How does the presence of singular points affect the Laplace spectrum? What happens to the spectrum when we pass form a smooth manifold to an orbifold by introducing some singular points? Or, conversely, what happens to the spectrum when we delete some singular points?

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- In Farsi-Proctor-Seaton '14 We introduce the Γ-Extension of the Spectrum of the Laplacian of a Riemannian Orbifold, where Γ is a finitely generated discrete group.
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Hodge Theory for Orbifolds with Boundary

Important middle step in the proof of our results on connected sums is the Hodge Decomposition for Orbifolds with Boundary.

First some terminology. Let M be an orbifold with a manifold boundary. Its Laplacian Δ on funsctions (and also on p-forms) is defined to be

$$\Delta = d\delta + \delta d,$$

where $\delta = (-1)^{np+n+1} * d*$ is the formal adjoint of d. (We actually take a suitable standard L^2-closure of these operators.)

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A differential form ω restricted to the boundary can be decomposed into tangential and normal components

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- Corresponding to Absolute Boundary Conditions (normal boundary component equal to zero) and the Relative Boundary Conditions (tangential boundary component equal to zero) we take a suitable Laplacian.
- Specifically, define d_c and δ_c by

$$Dom(d_c) := \{\omega \in C_0^\infty \Lambda^p(M)\},$$
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- We want knowledge about the eigenvalues of the Laplacian acting on functions under collapsings of disks of orbifolds, and connected sums.
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Work in Progress

- Generalize the above convergence results to the orbifold spectrum on p-forms, $\forall p$.