Riemannian submersions between Riemannian Lie groupoids

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Overview

Technical results:

- Riemannian groupoids $\Longrightarrow$ linearization of groupoids (GroupoidFest 2013)
- Riemannian submersions between Riemannian groupoids $\Longrightarrow$ linearization of submersions

Aplications:

- invariance under Morita equivalence of metrics ($\Longrightarrow$ metrics on stacks)
- Rigidity of proper groupoids
- Ehresmann Theorem for groupoids/stacks

References:

- M. del Hoyo & RLF, Riemannian metrics on Lie groupoids, *Journal für die reine und angewandte Mathematik (Crelle)*, 2015.
- M. del Hoyo & RLF, Fibrations and metrics on Lie groupoids, to be posted soon on the `arXive`.

Riemannian n-metrics on Lie groupoids

Nerve of a Lie groupoid:

$$\cdots G^n \rightrightarrows \cdots G^2 \rightrightarrows G^1 \rightrightarrows G^0 [\longrightarrow G^0/G^1]$$

Definition

A **n-metric** on G is a metric $\eta^{(n)}$ on the manifold G^n which is invariant under S_{n+1} and that is transverse to one (and hence all) face-map $\varepsilon_i : G^n \to G^{n-1}$. When $n = 2$, the pair $(G, \eta^{(2)})$ is called a **Riemannian groupoid**.

Remarks:

- n-metric $\Longrightarrow$ k-metric for all $n > k \geq 0$;
- 1-metric $\Longrightarrow$ orbit foliation is a (singular) Riemannian foliation;
- 3-metric $\Longrightarrow$ (unique) k-metric for all k.
- In general, the metric $\eta^{(1)}$ does not determine the metric $\eta^{(2)}$.

Examples of Riemannian groupoids

Many classes of groupoids are Riemannian:

1. **Unit groupoid:** $M \rightrightarrows M$ for any Riemannian manifold (M, η);
2. **Pair groupoid:** $M \times M \rightrightarrows M$ for any Riemannian manifold (M, η);
3. **Fibration groupoid:** $E \times_B E \rightrightarrows E$ for any Riemannian submersion $p : (E, \eta_E) \to (B, \eta_B)$;
4. **Holonomy groupoid:** $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ for any Riemannian foliation $(\mathcal{F}, \eta_M)$;
5. **Isometric group actions** $G \ltimes M \rightrightarrows M$ (complicated metrics constructed out of left-invariant metric η_G and η_M.
6. **Lie groups** (special case of isometric actions).

Theorem (del Hoyo & RLF (2015))

Proper groupoids $G \rightrightarrows M$ are Riemannian.

Proof: Multiple averaging arguments using Haar systems and quasi-actions. □

Linear model of a groupoid

$G \rightrightarrows M$ – Lie groupoid
$S \subset M$ – saturated neighborhood (=union of orbits of G)

$$\begin{tikzcd} \nu(G_S) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \nu(S) \arrow[d] \\ G_S \arrow[r, shift left] \arrow[r, shift right] & S \end{tikzcd}$$

Definition

1. The groupoid $\nu(G_S) \rightrightarrows \nu(S)$ is called the **linear model** around S
2. A **groupoid neighborhood** of $G_S \rightrightarrows S$ is a pair of opens $S \subset U$, $G_S \subset \tilde{U}$ such that $\tilde{U} \rightrightarrows U$ is a subgroupoid of $G \rightrightarrows M$,

Linearization Theorem

Theorem (del Hoyo, RLF (2015))

Let $G \rightrightarrows M$ be a Lie groupoid with a multiplicative metric $\eta^{(2)}$. Then G is weakly linearizable around any saturated $S \subset M$: there are groupoid neighborhoods of $G_S \rightrightarrows S$ in $G \rightrightarrows M$ and in $\nu^*(G_S) \rightrightarrows \nu^*(S)$ which are isomorphic.

Corollary

All known linearization theorems for Lie groupoids.

Proof of the Linearizarion Theorem:

$$\begin{tikzcd} \nu(G_S)^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\exp_{\eta^{(2)}}"'] & \nu(G_S) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\exp_{\eta^{(1)}}"'] & \nu(S) \arrow[d, "\exp_{\eta^{(0)}}"'] \\ G^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

$\square$

Remarks:

- Linearization of **proper** Lie groupoids was conjectured by Weinstein (2002) and proved by Zung (2005), using Banach space techniques. A more geometric proof, using groupoid cohomology, was given by Crainic & Struchiner (2011). Our result is more general and has an easier proof!
- Linearization Theorem $\Longrightarrow$ classical results:
  1. Ehresmann theorem: every proper submersions is locally trivial;
  2. Bochner Linearization Theorem/Slice Theorem: local model for a proper action around an orbit.
  3. Local Reeb Stability: local model for a foliation around compact leaf with finite holonomy.
- Linearization Theorem $\Longrightarrow$ new results:
  1. Local normal forms for Poisson manifolds around symplectic leaves (Marcut (2012), Crainic, RLF & Martinez-Torres (unpublished));

Riemannian submersions of Lie groupoids

Definition

A Lie groupoid map $\phi : \tilde{G} \to G$ is a called a **Riemannian submersion** if both $\tilde{G}$ and G are endowed with 2-metrics, for which $\phi^{(2)} : \tilde{G}^{(2)} \to G^{(2)}$ becomes a Riemannian submersion.

Remarks:

- $\phi^{(2)}$ Riemannian submersion $\Longrightarrow$ $\phi^{(1)}$ and $\phi^{(0)}$ Riemannian submersions;
- $\phi : \tilde{G} \to G$ is a *Riemannian k-submersion* if there exist k-metrics and $\phi^{(k)}$ is a Riemannian submersion;
- Riemannian k-submersion $\Longrightarrow$ Riemannian $(k-1)$-submersion.

Fibrations and cleavages

Are there many examples of Riemannian submersions?

Definition

A **fibration** is a map $\phi : \tilde{G} \to G$ such that $\phi^{(0)} : \tilde{M} \to M$ and

$$\phi' : \tilde{G} \to G \times_M \tilde{M}, \quad g \mapsto (\phi^{(1)}(g), s(g))$$

are surjective submersions. G is called the **base** and $\tilde{G}$ the **total groupoid**.

Definition

A **cleavage** for a fibration $\phi : \tilde{G} \to G$ is a smooth section:

$$\begin{tikzcd} \tilde{G} \arrow[r, "\phi'"] & G \times_M \tilde{M} \arrow[l, bend left, dashed, "\sigma"] \end{tikzcd}$$

A cleavage is call **unital** if it preserves identities and **flat** if it is closed under multiplication. A **split fibration** is a fibration with a unital, flat, cleavage.

Examples of fibrations and cleavages

- a fibration between unit groupoids $\phi : \tilde{M} \to M$ is the same as a surjective submersion and it is always split: it has a unique cleavage which is flat and unital;
- A fibration between Lie groups $\phi : \tilde{G} \to G$ is the same as a surjective Lie group homomorphism; a cleavage $\sigma : G \to \tilde{G}$ is simply a smooth section and it may be split or not.
- A fibration $\phi : (\tilde{G} \rightrightarrows \tilde{M}) \to (M \rightrightarrows M)$ is called a **family of Lie groupoids** parametrized $M \iff$ surjective submersion $\tilde{M} \to M$ constant along the $\tilde{G}$-orbits. It has an obvious flat unital cleavage, so it is always split.
- For a Lie groupoid action $G \curvearrowright \tilde{M}$ along a submersion $\mu : \tilde{M} \to M$ the projection $(G \ltimes \tilde{M} \rightrightarrows \tilde{M}) \to (G \rightrightarrows M)$ is a fibration. Every fibration for which ϕ' is bijective arises in this way and ϕ is then called an **action fibration**. It has a unique flat unital cleavage, so it is always split.
- If $\phi : \tilde{G} \to G$ is a Morita map and $\phi^{(0)}$ is a surjective submersion, then ϕ is a fibration, called a **Morita fibration**. It may admit a cleavage or not (e.g., the usual covering $\mathbb{S}^1 \to \mathbb{R}$ gives a Morita fibration between the pair groupoids which has no cleavage).

Split Fibrations

There is a 1:1 correspondence:

$$\boxed{\{\text{split fibrations}\} \quad \longleftrightarrow \quad \begin{array}{c}\text{semi-direct products}\\ (G \times_M K \rightrightarrows \tilde{M}) \to (G \rightrightarrows M)\end{array}}$$

where $G \rightrightarrows M$ acts on a family of Lie groupoids $q : (K \rightrightarrows \tilde{M}) \to M$.

Theorem

Every split fibration $\phi : \tilde{G} \to G$ between proper groupoids can be made Riemannian, i.e., there exist 2-metrics $\tilde{\eta}$ on $\tilde{G}$ and η on G for which ϕ is a Riemannian submersion.

Proof: Averaging adapted to the fibration. □

Linear model of a submersion

For a submersion $\phi : \tilde{G} \to G$ and a saturated submanifold $S \subset M$:

- $\tilde{S} = \phi^{-1}(S) \subset \tilde{M}$ is a saturated submanifold
- induced map between the linear local models:

$$\overline{\mathrm{d}\phi} : (\nu(\tilde{G}_{\tilde{S}}) \rightrightarrows \nu(\tilde{S})) \to (\nu(G_S) \rightrightarrows \nu(S)).$$

Definition

We say that ϕ is **linearizable around** S if there is a linearization $\tilde{\alpha}$ of $\tilde{G}$ around $\tilde{S} = \phi^{-1}(S)$ and a linearization α of G around S inducing a commutative square:

$$\begin{tikzcd} \nu(\tilde{G}_{\tilde{S}}) \supset \tilde{U} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\overline{\mathrm{d}\phi}"'] & \tilde{V} \subset \tilde{G} \arrow[d, "\phi"] \\ \nu(G_S) \supset U \arrow[r, "\alpha"] & V \subset G \end{tikzcd}$$

Linearization of Riemannian submersions

Theorem (del Hoyo & RLF (2015))

Let $\phi : (\tilde{G}, \tilde{\eta}) \to (G, \eta)$ a Riemannian submersion between Lie groupoids, let $S \subset M$ embedded saturated, and let $\tilde{S} = \phi^{-1}(S)$. Then the exponential maps of $\tilde{\eta}$ and η define a linearization of ϕ around S.

Corollary

A split fibration between proper Lie groupoids is linearizable.

Ehresman Theorem for groupoid fibrations

Theorem

Let $\phi : \tilde{G} \to G$ be a proper split fibration between proper Lie groupoids, and let $S \subset M$ be a saturated submanifold. Then there exist open subgroupoids $G_S \subset U \subset \nu(G_S)$ and $G_S \subset V \subset G$, and linearization maps:

$$\begin{tikzcd}
\nu(\tilde{G}_{\tilde{S}}) \supset (\overline{\mathrm{d}\phi})^{-1}(U) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\overline{\mathrm{d}\phi}"'] & \phi^{-1}(V) \subset \tilde{G} \arrow[d, "\phi"] \\
\nu(G_S) \supset U \arrow[r, "\alpha"] & V \subset G
\end{tikzcd}$$

Deformations of Lie groupoids

Definition

A **deformation** of $G \rightrightarrows M$ parametrized by I with base point $0 \in I$ consists of a fibration $\phi : \tilde{G} \to I$ such that $\phi^{(1)}, \phi^{(0)}$ are locally trivial and the central fiber $\tilde{G}_0$ is isomorphic to G. The deformation is called:

- *k*-**parameter deformation** if $I \subset \mathbb{R}^k$ open subset;
- **proper** if ϕ is a proper map;
- **trivial** if it is isomorphic to the product family $G \times I \to I$;

Remarks:

1. A groupoid admits proper deformations only when it is compact;
2. For proper deformation, the conditions that $\phi^{(1)}$ and $\phi^{(0)}$ be locally trivial are automatic;
3. In a deformation, one can think that the manifolds G and M remain fixed while, when the parameter $\varepsilon \in I$ varies, one deforms the structure maps $s_\varepsilon, t_\varepsilon, m_\varepsilon, u_\varepsilon, i_\varepsilon$ of the groupoid G.

Examples of deformations

1. (deformation of a Lie group) Let $G =]0, +\infty[\times \mathbb{R}$ and consider a family of groups $\phi : G \times \mathbb{R} \to \mathbb{R}$, $(x, y, \varepsilon) \mapsto \varepsilon$, with multiplication:
$$(x_1, y_1) \star_\varepsilon (x_2, y_2) := (x_1 + x_2, y_1 + (x_1)^\varepsilon y_2).$$
2. (deformation of a group action) Consider the family of group actions of $\mathbb{R}$ on $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by:
$$x \star_\varepsilon (\theta_1, \theta_2) = (\theta_1 + \varepsilon x, \theta_2 + \varepsilon x).$$
with associated fibration $\phi : G \times \mathbb{R} \to \mathbb{R}$ where $G = \mathbb{R} \times \mathbf{T}^2 \rightrightarrows \mathbf{T}^2$.
3. (deformation of a foliation) Consider the family of foliations $\mathcal{F}_\varepsilon$ of $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by the distributions:
$$T\mathcal{F}_\varepsilon = \left(\frac{\partial}{\partial \theta_1} + \varepsilon \frac{\partial}{\partial \theta_2} \right).$$
The corresponding monodromy groupoids $\mathrm{Mon}(\mathcal{F}_\varepsilon) = \mathbf{T}^2 \times \mathbb{R} \rightrightarrows \mathbf{T}^2$ give a 1-parameter deformation $\phi : G_0 \times \mathbb{R} \to \mathbb{R}$, $(\theta_1, \theta_2, x, \varepsilon) \to \varepsilon$, of $G_0 = \mathrm{Mon}(\mathcal{F}_0)$.

Compact Lie groupoids are rigid

Theorem

Compact Lie groupoids are rigid: a proper deformation $\tilde{G} \to I$ of a compact Lie groupoid $G \rightrightarrows M$ is locally trivial.

Proof. For G be a compact Lie groupoid the family

$$\phi : (\tilde{G} \rightrightarrows \tilde{M}) \to (\mathbb{R}^k \rightrightarrows \mathbb{R}^k)$$

is linearizable. The local linear model around the central fiber is just the trivial family $G \times \mathbb{R}^k \times \mathbb{R}^k$. □

Compact Lie groupoids are rigid

Some consequences of the rigidity theorem:

- the rigidity of Lie group structures on a compact manifold;
- the rigidity of smooth actions $K \curvearrowright M$ of a fixed compact group on a fixed compact manifold (Palais).
- the rigidity of compact Hausdorff foliations of M whose leaves have finite fundamental group (Rosenberg *et al.*)

Remark: Similar results were obtained recently by Crainic, Mestre and Struchiner, using completely different techniques (deformation cohomology). Preprint `arXiv:1510.02530`.

Rigidity of proper Lie groupoids

Proper Lie groupoids fail to be rigid:

Example

Let $\mathbb{Z}_2$ act on $\mathbb{R}$ by reflection in the origin and trivially on an exotic $\mathbb{R}^4_e$. The induced diagonal action $\rho : \mathbb{Z}_2 \curvearrowright \mathbb{R} \times \mathbb{R}^4_e \simeq \mathbb{R}^5$ is a non-linear action that is not isomorphic to the linearized action at the origin: the fixed point sets of the linear action (i.e., $\mathbb{R}^4$) and of the non-linear action (i.e., $\mathbb{R}^4_e$) are not diffeomorphic. Hence, the family of $\mathbb{Z}_2$-actions $\rho_\varepsilon(x) := 1/\varepsilon\rho(\varepsilon x)$ yields a non-trivial 1-parameter deformation $\phi : G \times \mathbb{R} \to \mathbb{R}$ of the proper groupoid $G = \mathbb{Z}_2 \ltimes \mathbb{R}^5 \rightrightarrows \mathbb{R}^5$.

Theorem

A k-parameter deformation $G \times \mathbb{R}^k \to \mathbb{R}^k$ of a proper groupoid G that fixes the source and target is trivial.

Other applications and on-going work:

Other applications:

- Invariance under Morita equivalence of groupoid metrics;
- Ehresman Theorem for stacks.

On-going work and future directions:

- Harmonic representatives for equivariant cohomology;
- Hodge Theory for geometric stacks;
- Singularities of geometric flows.