A Stabilization Theorem for Fell Bundles over Groupoids (part 1)

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Groupoid Fest 2015, University of Memphis

United States Naval Academy

Fell bundles

Definition

A Fell bundle $p : \mathcal{B} \to G$ over a locally compact Hausdorff groupoid G is an upper semicontinuous Banach bundle equipped with a multiplication map $(a, b) \to ab$ and an involution $b \to b^*$ such that

1. $p(ab) = p(a)p(b)$,
2. $p(b^*) = p(b)^{-1}$,
3. $(ab)^* = b^* a^*$,
4. for each $u \in G^{(0)}$, $B(u)$ is a C^*-algebra,
5. for each $x \in G$, $B(x)$ is a $B(r(x)) - B(s(x))$-imprimitivity bimodule with the inner products

$$_{B(r(x))}\langle a, b\rangle = ab^* \text{ and } \langle a, b\rangle_{B(s(x))} = a^* b.$$

Example: dynamical systems

Example

- Assume that (A, G, α) is a dynamical system with G a group.
- $\mathcal{B}$ is the trivial bundle $A \times G$
- The multiplication in $\mathcal{B}$ is given by

$$(a, s)(b, t) = (a\alpha_s(b), st)$$

and

$$(a, s)^* = (\alpha_s^{-1}(a^*), s^{-1}).$$

Example: groupoid dynamical systems

Example

Let $(\mathcal{A}, G, \alpha)$ be a groupoid dynamical system. Then

$$\mathcal{B} := \mathcal{A} \rtimes_\alpha G := r^*\mathcal{A} = \{(a, x) : \pi(a) = r(x)\}$$

with the operations

$$(a, x)(b, y) := (a\alpha_x(b), xy)$$

and

$$(a, x)^* := (\alpha_x^{-1}(a^*), x^{-1})$$

is a Fell bundle over G.

C^*-algebra of a Fell bundle

Definition

Assume that G has a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$. For $f, g \in \Gamma_c(G; \mathcal{B})$ one defines

$$f * g(x) = \int_G f(y) g(y^{-1}x) d\lambda^{r(x)}(y)$$

and

$$f^*(x) = f(x^{-1})^*.$$

$C^*(G; \mathcal{B})$ is the completion of $\Gamma_c(G; \mathcal{B})$ with respect to the *universal* norm.

Main Result

Theorem (Kumjian '98; Muhly '00; Kumjian, Sims, Williams and M.I., 15)

Let $p : \mathcal{B} \to G$ be a second countable, saturated Fell bundle over a locally compact Hausdorff groupoid G endowed with a Haar system $\{\lambda_x\}_{x \in G^{(0)}}$. Then there is a groupoid dynamical system $(\mathcal{K}, G, \alpha)$ such that $C^*(G; \mathcal{B})$ and $C^*(\mathcal{K} \rtimes_\alpha G)$ are Morita equivalent and so are $C^*_{red}(G; \mathcal{B})$ and $C^*_{red}(\mathcal{K} \rtimes_\alpha G)$.

Construction of the groupoid dynamical system

Theorem

Let $p : \mathcal{B} \to G$ be a Fell bundle over G. For $x \in G^{(0)}$ let $V(x) = L^2(G_x; \mathcal{B}, \lambda_x)$. Then $V(x)$ is a full right $A(x)$-Hilbert module. Moreover, if $V := \bigsqcup_{x \in G^{(0)}} V(x)$ and $\nu : V \to G^{(0)}$ is the projection map, then $\nu : V \to G^{(0)}$ is an upper semicontinuous Banach bundle over $G^{(0)}$ and $\mathcal{V} := \Gamma_0(G^{(0)}, V)$ is a full Hilbert A-module.

Construction of the C^*-bundle

Theorem

Let $p : \mathcal{B} \to G$ be a Fell bundle over G and let V be bundle over $G^{(0)}$ from the previous slide. Then there is an upper semicontinuous C^*-bundle $k : \mathcal{K}(V) \to G^{(0)}$ such that there is a $C_0(G^{(0)})$ linear isomorphism of $\mathcal{K}(\mathcal{V})$ onto $\Gamma_0(G^{(0)}, \mathcal{K}(V))$ and such that $\mathcal{K}(V)(x)$ is isomorphic to $\mathcal{K}(V(x))$.

The action

Theorem

For $g \in G$, the map $\beta_g : V(r(g)) \otimes_{A(r(g))} B(g) \to V(s(g))$ defined on elementary tensors by

$$\beta_g(\xi \otimes b)(\gamma) = \xi(\gamma g^{-1})b$$

extends to an isometric isomorphism of Hilbert $A(s(g))$-modules. Then the map α_g defined by

$$\alpha_g(\beta_g(\xi \otimes b) \otimes \eta^*) = \xi \otimes \beta_{g^{-1}}(\eta \otimes b^*)^*$$

extends to a $*$-isomorphism between $\mathcal{K}(V(s(g)))$ and $\mathcal{K}(V(r(g)))$. Moreover, $(\mathcal{K}(V), G, \alpha)$ is a groupoid dynamical system.

The equivalence

Theorem

For $g \in G$ let $E(g) = V(r(g)) \otimes_{A(r(g))} B(g)$, let $\mathcal{E} = \bigsqcup_{g \in G} E(g)$, and let $q : \mathcal{E} \to G$ be the projection map. Then $q : \mathcal{E} \to G$ is an upper semicontinuous Banach bundle over G and a $\mathcal{K}(V) \rtimes_\alpha G - \mathcal{B}$ equivalence. Hence $C^*(G; \mathcal{B})$ and $C^*(G; \mathcal{K}(V) \rtimes_\alpha G)$ are Morita equivalent and so are $C^*_{red}(G; \mathcal{B})$ and $C^*_{red}(G; \mathcal{K}(V) \rtimes_\alpha G)$.

Applications

Theorem

Let G be a Hausdorff locally compact groupoid and let $p : \mathcal{B} \to G$ be a continuous Fell bundle. Let A be the C^*-algebra over $G^{(0)}$. Assume that the action of G on $\operatorname{Prim} A$ is amenable and essentially free. Then the lattice of ideals of $C^*(G; \mathcal{B})$ is isomorphic to the lattice of invariant open sets of $\operatorname{Prim} A$.

Theorem (Kumjian, Pask, Sims, 2014; Kumjian, Sims, Williams, and M.I., 2015)

Under the same hypothesis, $C^*(G; \mathcal{B})$ is simple if and only if the action of G on $\operatorname{Prim} A$ is minimal.