Cohomology for categories, k-graphs, and groupoids

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Cohomology for groupoids and k-graphs

Many flavors of groupoid cohomology:

- Borel cocycle cohomology [Moo64]
- Continuous cocycle cohomology [Ren80]
- Sheaf cohomology [Kum88]
- Brauer group [KWMR98]
- Cohomology from simplicial spaces [Hae79, Moe98, Tu06]

For a higher-rank graph Λ, we have

- Cubical cohomology of Λ [KPS12]
- Categorical cohomology of Λ [KPS15]
- Cohomology for small categories [Xu08, BW, Wat66]

Why another one?

- To a k-graph Λ, associate (functorially) a groupoid $\mathcal{G}_\Lambda$
  Do we get maps from k-graph cohomology to groupoid cohomology?
- [KPS15] construct maps $H^i(\Lambda, A) \to H^i_c(\mathcal{G}_\Lambda, A)$ for $i \leq 2$
  Very different constructions for $i = 0, 1, 2$!
  Don't generalize readily to $i > 2$
- Cohomology for small categories fixes this

Outline

- Interpret cohomology for small categories in terms of higher-rank graphs Λ
  - Λ-modules $\mathcal{A}$
- Define a functor $\mathcal{A} \mapsto \underline{\mathcal{A}}$ from Λ-modules to $\mathcal{G}_\Lambda$-sheaves
- Obtain a map $\psi^n : H^n(\Lambda, \mathcal{A}) \to H^n_c(\mathcal{G}_\Lambda, \underline{\mathcal{A}})$ $\underline{\text{for all } n}$
- Future work: Connect ψ^2 with $\sigma : H^2(\Lambda, A) \to H^2_c(\mathcal{G}_\Lambda, A)$ of [KPS15]

Higher-rank graphs

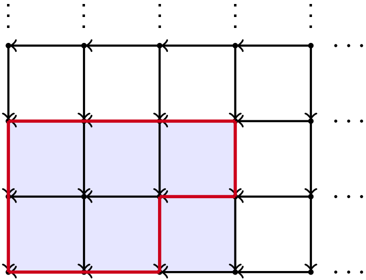
What are k-graphs?

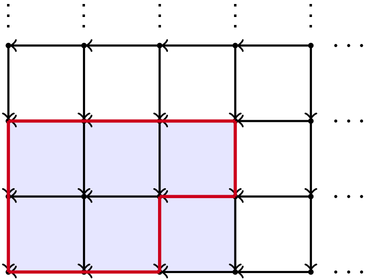
- Directed graph $\rightsquigarrow$ k-graph Λ
- Paths $\rightsquigarrow$ k-dimensional rectangles
- Degree functor $d : \Lambda \to \mathbb{N}^k$ tells us the size of the "path"

Higher-rank graphs

Definition

A k-graph is a countable category Λ with a degree map $d : \Lambda \to \mathbb{N}^k$ satisfying the following factorization property: Whenever $\lambda \in \Lambda$ has $d(\lambda) = m + n$, $\exists! \ \mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$.





In our example,

$$d(\lambda) = (3, 2) = (0, 2) + (3, 0) = (2, 0) + (0, 1) + (1, 0) + (0, 1),$$

so each of these possible factorizations must give us the same element λ.

Infinite paths in k-graphs

Write Λ^∞ for the space of infinite paths in Λ.

For each $p \in \mathbb{N}^k$, set $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by "chopping off" the initial segment of shape p.

More formally: Let

$$\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}.$$

This is a k-graph with $d(m, n) = n - m$.

An infinite path $x \in \Lambda^\infty$ is a degree-preserving functor $x : \Omega_k \to \Lambda$.

$$\sigma^p(x)(m, n) = x(m + p, n + p)$$

Think of x as a way to map $\mathbb{N}^k$ into Λ; $\sigma^p(x)$ corresponds to the isomorphism $\mathbb{N}^k \longleftrightarrow \mathbb{N}^k + p$.

k-graph groupoids

$$\mathcal{G}_\Lambda := \{(x, p-q, y) : x, y \in \Lambda^\infty, \sigma^p(x) = \sigma^q(y)\}$$

$$\mathcal{G}_\Lambda^{(0)} \cong \Lambda^\infty = \{(x, 0, x) : x \in \Lambda^\infty\}$$

$$(x, n, y)(y, \ell, z) = (x, n+\ell, z)$$

Sheaves and Λ-modules

Definition

A Λ-module is a contravariant functor $\mathcal{A} : \Lambda \to \mathbf{Ab}$.

To each vertex (object) $v \in \Lambda^0$, associate $A_v \in \mathbf{Ab}$. To an element (morphism) $\lambda : v \to w$ of Λ, associate $A(\lambda) : A_w \to A_v$ (group homomorphism).

A $\underline{\mathcal{G}\text{-sheaf}}$ $\mathscr{A}$ consists of:

- An abelian group $\mathscr{A}(x)$ at each $x \in \mathcal{G}^{(0)}$
- An isomorphism $\alpha_\gamma : \mathscr{A}(s(\gamma)) \to \mathscr{A}(r(\gamma))$ for each $\gamma \in \mathcal{G}$
- A topology on $\mathscr{A} = \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathscr{A}(x)$ making everything continuous and consistent.

From Λ-modules to $\mathcal{G}_\Lambda$-sheaves

Let $\mathcal{A}$ be a Λ-module. For $x \in \Lambda^\infty$, define

$$\underline{\mathcal{A}}(x) = \varinjlim_{\Omega_k} (\mathcal{A}_{x(p)}, \mathcal{A}(x(p,q)))$$

Ω_k is a directed system: Given any two rectangles $(m,n), (p,q) \in \Omega_k$, there is a rectangle $(M,N) \in \Omega_k$ containing both. $M_i = \min\{m_i, p_i\}$; $N_i = \max\{n_i, q_i\}$.
Recall that $x(p,q) \in \Lambda$ is a morphism with range $x(p)$, source $x(q)$; $\mathcal{A}$ is a <u>contravariant</u> functor.
$\mathcal{G}_\Lambda$-action?
Let $\varphi_p^x : \mathcal{A}_{x(p)} \to \underline{\mathcal{A}}(x)$. Then

$$(x, p-q, y) \cdot (\varphi_q^y(a)) = \varphi_p^x(a).$$

Theorem (G-Kumjian)

*The map $\mathcal{A} \mapsto \underline{\mathcal{A}}$ is a functor from Λ-**mod** to $\mathcal{G}_\Lambda$-**shf**.*

Continuous cohomology from sheaves

Usually, one defines an A-valued n-cochain on $\mathcal{G}$ as a function $f : \mathcal{G}^{(n)} \to A$. We have a coboundary δ^n taking n-cochains to $(n-1)$-cochains.

$$H_c^n(\mathcal{G}, A) = \frac{\ker \delta^n}{\operatorname{Im} \delta^{n+1}}$$

Alternatively:
From $\mathcal{G}^{(n)}$, construct a relative projective resolution $\mathscr{P}_*$ of the trivial $\mathcal{G}$-sheaf $\underline{\mathbb{Z}}$.

Proposition (G-Kumjian)

$$H_c^n(\mathcal{G}, \mathscr{A}) = H^n(\operatorname{Hom}(\mathscr{P}_*, \mathscr{A})).$$

From k-graph cohomology to groupoid cohomology

$$F^n_v = \mathbb{Z}\{(\lambda_1, \ldots, \lambda_n) \in v\Lambda^{*n} : s(\lambda_n) = v\}$$

$$F^n(\lambda)[\lambda_1, \ldots, \lambda_n] = [\lambda_1, \ldots, \lambda_n\lambda]$$

We define

$$H^n(\Lambda, \mathcal{A}) = H^n(\mathrm{Hom}(F^*, \mathcal{A})).$$

Theorem (G-Kumjian)

$\underline{F^*}$ is a relative projective resolution of the trivial $\mathcal{G}_\Lambda$-sheaf $\mathbb{Z}$.

From k-graph cohomology to groupoid cohomology

Theorem (G-Kumjian)

$$H^n(\mathrm{Hom}(\mathscr{P}_*, \mathscr{A})) \cong H^n(\mathrm{Hom}(\underline{F^*}, \mathscr{A})).$$

$$H^n(\Lambda, \mathcal{A}) \to H^n(\mathrm{Hom}(\underline{F^*}, \underline{\mathcal{A}})) \xleftarrow{\cong} H^n_c(\mathcal{G}_\Lambda, \underline{\mathcal{A}}).$$

To compare our map $\psi^2 : H^2(\Lambda, A) \to H^2(\mathcal{G}_\Lambda, A)$ with σ of [KPS15], we'll have to "flip arrows."

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