The blow-up construction of Lie groupoids

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Outline

Elementary modification of Lie algebroids Elementary modification of Lie algebroids Examples

The blow-up construction of Lie groupoids

Blow-up of smooth manifolds Blow-up of Lie groupoids

Elementary modification of vector bundles

- ► *M* is a manifold, and *L* is a closed hypersurface.
- A is a vector bundle over M, and B is a subbundle of $A|_L$.

The elementary modification of A along B is the vector bundle [A:B] over M with the space of sections

$$\Gamma(M, [A:B]) = \{ X \in \Gamma(M, A) \mid X|_L \in \Gamma(L, B) \}.$$
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Elementary modification of Lie algebroids

If A is a Lie algebroid, and B is a Lie subalgebroid, then [A:B] is a Lie algebroid.

Example 1: Log tangent bundle

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If $M = \mathbb{R}^2$ and $L = \{x = 0\}$ is the y-axis, then $T\mathbb{R}^2(-\log L)$ has a basis

$$x\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}$$

Example 2: Log symplectic structure

- dim M = 2n.
- A log symplectic structure on (*M*, *L*) is a Poisson structure π which is nondegnerate on *M* \ *L* and degenerates linearly on *L*.
- ► The hypersurface *L* is foliated by 2n 2 dimensional symplectic leaves. Let $T_{\pi}L$ be the foliated tangent algebroid.
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- ► As a Lie algebroid, the Poisson algebroid T_{π}^*M is isomorphic $[TM: T_{\pi}L]$.
- If $M = \mathbb{R}^2$ and $\pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, then we have

$$\langle dx, dy
angle = T_{\pi}^* M \cong [TM: T_{\pi}L] = \langle x \frac{\partial}{\partial y}, -x \frac{\partial}{\partial x}
angle.$$

Integration?

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Answer: The blow-up construction of Lie groupoids.

- X is a manifold.
- Y is a submanifold of codimension at least 2.
- The blow-up of X along Y, denoted by Bl_Y(X), is a smooth manifold obtained by replacing Y ⊂ X with the exceptional divisor, which is isomorphic to the projectivization of the normal bundle NY.
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If $X = \mathbb{R}^2$ and Y = 0 is the origin, then $Bl_0(\mathbb{R}^2)$ is isomorphic to $\mathbb{R}P^2$ minus a point.

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Figure : $Bl_0(\mathbb{R}^2)$

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- ► X is a manifold, and Y is a submanifold of codimension at least 2.
- $Z \subset X$ is a submanifold that intersects Y transversely.
- The proper transform of Z, denoted by \overline{Z} , is defined as follows:

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If $X = \mathbb{R}^2$, Y = 0 is the origin, and $Z = \{y = 0\}$ is the x-axis, then Z and $\overline{\overline{Z}}$ are the red lines in the following pictures.



Figure : \mathbb{R}^2 and the x-axis *Z*



Figure : $Bl_0(\mathbb{R}^2)$ and the proper transform $\overline{\overline{Z}}$

Blow-up of Lie groupoids

- ► *M* is a manifold, and *L* is a closed hypersurface.
- $G \rightrightarrows M$ is a Lie groupoid.
- $H \rightrightarrows L$ is a subgoupoid.
- We define

$$[G:H] = \mathsf{Bl}_{H}(G) \setminus \left(\overline{\overline{s^{-1}(L)}} \cup \overline{\overline{t^{-1}(L)}}\right).$$
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We equip [G: H] with a groupoid structure over M such that p: [G: H] → G is a groupoid morphism. That is,

$$\widetilde{s}: [G:H] \to M$$

is the composition of the blow-down map $p : [G:H] \to G$ and $s : G \to M$, and similarly $\tilde{t} = p \circ t$.

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Blow-up of Lie groupoid

Theorem (Gualtieri-Li)

If $G \Rightarrow M$ is a Lie groupoid with Lie algebroid A, and $H \Rightarrow L$ is a subgoupoid over a hypersurface $L \subset M$ with Lie algebroid B, then $[G:H] \Rightarrow M$ is a Lie groupoid, and the Lie algebroid of [G:H] is [A:B]. That is,

$$\operatorname{Lie}\left([G:H]\right) = [\operatorname{Lie}(G):\operatorname{Lie}(H)]. \tag{2.2}$$

Example 1: Log tangent bundle

The Lie groupoid integrating the log tangent algebroid $TM(-\log L) = [TM:TL]$ is given by

$$[M \times M : L \times L] \subset \mathsf{Bl}_{L \times L}(M \times M)$$

where $M \times M \rightrightarrows M$ and $L \times L \rightrightarrows L$ are the pair groupoids.

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If we think of \mathbb{R}^2 as the pair groupoid over \mathbb{R} and the origin $0 \in \mathbb{R}^2$ as the pair groupoid over $0 \in \mathbb{R}$, then $[\mathbb{R}^2:0]$ is illustrated as follows:



Figure : \mathbb{R}^2 as the pair groupoid over \mathbb{R}



Figure : The groupoid $[\mathbb{R}^2\!:\!0]$ where the red lines are removed

Example 2: Log symplectic manifold

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Hence, the symplectic groupoid of (M, π) is

$$[M \times M : L \times_{S^1} L]$$

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Thank you!