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## The discriminant invariant of Cantor group actions

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### Introduction

Let  $\boldsymbol{X}$  be a Cantor set, and  $\boldsymbol{G}$  be a finitely generated non-abelian group.

Let  $\Phi: G \to Homeo(X)$ , or  $(X, G, \Phi)$  be a minimal action of G on X.

#### Problem

Given a group G, classify minimal actions  $(X, G, \Phi)$ .

In this talk:

- discuss the classification of equicontinuous minimal actions on Cantor sets by the *degree of homogeneity*.
- introduce the invariant, called the *discriminant group*, which is related to the degree of homogeneity.

J. Dyer, Dynamics of equicontinuous group actions on Cantor sets, doctoral thesis, UIC, 2015.

J. Dyer, S. Hurder and O. Lukina, The discriminant invariant of Cantor group actions, arXiv:1509.06227.

# The degree of homogeneity

A homeomorphism  $h:X\to X$  is an automorphism of  $(X,G,\Phi)$  if h commutes with the G-action on X.

Let  $Aut(X,G,\Phi)$  be its group of automorphisms.

### Definition

A Cantor minimal system  $(X, G, \Phi)$  is:

- regular if the action of  $Aut(X, G, \Phi)$  on X is transitive;
- weakly regular if the action of Aut(X, G, Φ) decomposes X into a finite collection of orbits;
- *irregular* if the action of  $Aut(X, G, \Phi)$  decomposes X into an infinite collection of orbits.

## Equicontinuous actions

Let d be a metric on X.

The action  $(X,G,\Phi)$  is *equicontinuous* if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that for any  $g \in G$  and any  $x, y \in X$  with  $d(x,y) < \delta$  we have

 $d(\Phi(g)(x),\Phi(g)(y))<\epsilon.$ 

### Auslander, 1988

Let  $(X, {\cal G}, \Phi)$  be a regular Cantor minimal system, then it is equicontinuous.

#### Problem

Let  $(X,G,\Phi)$  be a minimal equicontinuous action. Find an invariant which allows to distinguish between regular, weakly regular and irregular actions.

### Group chains

Actions on Cantor sets are traditionally studied using *Kakutani-Rokhlin partitions*. We use a special type of such partitions, *coding partitions*.

#### Proposition

Given a minimal equicontinuous action  $(X,G,\Phi),$  where exists a family of partitions  $\{\mathcal{P}_i\}_{i\geq 0},$  such that

- For every  $U \in \mathcal{P}_i$  we have  $diam(U) < \frac{1}{2^i}$ .
- $\mathcal{P}_i = \{g \cdot V_i\}_{g \in G}$ , where  $V_i$  is a clopen set in X.
- The set  $G_i = \{g \in G \mid g \cdot V_i = V_i\}$  is a subgroup of finite index in G.
- We have  $V_{i+1} \subset V_i$ , and so the partition  $\mathcal{P}_{i+1}$  refines  $\mathcal{P}_i$ , and  $\{G_i\}_{i\geq 0}$  is a nested group chain.
- There is a homeomorphism  $\phi: X \to G_{\infty} = \lim_{\longleftarrow} \{G/G_i \to G/G_{i-1}\}$ which commutes with the action of G on X and  $G_{\infty}$ .

### Regular examples in the Heisenberg group

Let  $\mathcal{H}=(\mathbb{Z}^3,*)$  be the discrete Heisenberg group with operation \*,

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy').$$

Consider subgroups in the form  $\Gamma = M\mathbb{Z}^2 \times m\mathbb{Z}$ , where  $M = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ .

#### Lemma

A set  $\Gamma$  is a subgroup if and only if m divides both entries of one of the rows of M.

Lightwood, Şahin and Ugarcovici, 2014 give examples of group chains of normal subgroups  $\{N_i\}_{i\geq 0}$  in  $\mathcal{H}$ .

We will show that their examples give regular actions  $(N_{\infty}, G)$ , where

$$N_{\infty} = \lim_{\longleftarrow} \{ \mathcal{H}/N_i \to \mathcal{H}/N_{i-1} \}.$$

### Non-regular examples in the Heisenberg group

### Dyer, Hurder, Lukina, 2015:

Let  $\mathcal{H}$  be the discrete Heisenberg group. Let  $M_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix}$ , and  $A_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$ , where p and q are distinct primes. Then

• The action represented by a group chain  $\Gamma_0 = \mathcal{H}$ ,  $\{\Gamma_n\}_{n \ge 1} = \{M_n \mathbb{Z}^2 \times p\mathbb{Z}\}_{n \ge 1}$  is weakly regular and not regular.

**2** The action represented by a group chain 
$$G_0 = \mathcal{H}$$
,  $\{G_n\}_{n \ge 1} = \{A_n \mathbb{Z}^2 \times p^n \mathbb{Z}\}_{n \ge 1}$  is irregular.

How can one see that an action is regular, weakly regular or irregular?

# Uniqueness of group chains

Let (X, G) be a minimal equicontinuous group action.

#### Uniqueness of group chain representation

The group chain  $\{G_i\}_{i\geq 0}$  depends on the following:

- **()** The choice of the collection of partitions  $\{\mathcal{P}_i\}_{i\geq 0}$ ,  $\mathcal{P}_i = \{g \cdot V_i\}_{g\in G}$ .
- **2** The point  $x \in X$  such that  $\{x\} = \bigcap_i V_i$ .

How much the properties of groups in the chain depend on the choices?

**Rogers and Tollefson, 1971:** For  $n \ge 1$ , there exists a closed manifold  $M^n$  with fundamental group G, such that

$$G \supset G_1 \supset N_1 \supset G_2 \supset N_2 \supset \cdots$$

where  $|G:G_i| < \infty$  and  $|G:N_i| < \infty$ , and for each  $i \ge 0$  the group  $N_i$  is normal in G, while  $G_i$  is not normal in G.

Equivalences

### Equivalent chains

#### Question

Let  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  be group chains in G, and let  $(G_{\infty}, G)$  and  $(H_{\infty}, G)$  be the inverse limit systems. When are  $(G_{\infty}, G)$  and  $(H_{\infty}, G)$  conjugate?

Introduce the following notions:

Equivalent chains (Rogers and Tollefson, 1971)

The group chains  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  are *equivalent* if there exist subsequences  $\{G_{i_k}\}_{i\geq 0}$  and  $\{H_{i_k}\}_{i\geq 0}$  such that

 $G_0 \supset H_0 \supset G_{i_1} \supset H_{i_1} \supset \cdots$ 

The group chains  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  are *conjugate equivalent* if there exists  $(g_i) \in G$  such that  $\{g_i G_i g_i^{-1}\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  are equivalent.

# Pointed conjugacy of inverse limit systems

Let  $(X, G, \hat{x})$  be a minimal equicontinuous action with basepoint  $\hat{x} \in X$ .

#### Pointed conjugacy

The systems  $(X, G, \hat{x})$  and  $(Y, G, \hat{y})$  are *pointed conjugate* if and only if there exists a homeomorphism  $h: X \to Y$  such that  $h(\hat{x}) = \hat{y}$ , and for any  $x \in X \ g \cdot h(x) = h(g \cdot x)$ .

#### Theorem (Dyer, Hurder, Lukina, 2015)

Let  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  be group chains in G. The inverse limit systems  $(G_{\infty}, G)$  and  $(H_{\infty}, G)$  are pointed conjugate if and only if  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  are equivalent.

# Examples of equivalent chains

#### Different partitions

Let  $\{\mathcal{P}_i\}$  be partitions of X with group chain  $\{G_i\}_{i\geq 0}$  at x, and  $\{\mathcal{Q}_i\}$  be partitions of X with group chain  $\{H_i\}_{i\geq 0}$  at x. If  $diam(V) \to 0$  for all  $V \in \mathcal{P}_i$ , and  $diam(U) \to 0$  for all  $U \in \mathcal{Q}_i$ , then  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  are equivalent.

#### Automorphisms

Let  $\{\mathcal{P}_i\}_{i\geq 0}$  be partitions with group chain  $\{G_i\}_{i\geq 0}$  at x, and  $\{\mathcal{Q}_i\}_{i\geq 0}$  be partitions with group chain  $\{H_i\}_{i\geq 0}$  at y. Then there is  $h\in Aut(X,G,\Phi)$  such that h(x)=y if and only if  $\{G_i\}_{i\geq 0}$  and  $\{H_i\}_{i\geq 0}$  are equivalent.

## Equivalence relations on the set of group chains

Let G be a finitely generated group, and  $\mathfrak{G}$  be the set of all nested chains of all subgroups of finite index in G.

Given  $\{G_i\}_{i\geq 0}$ , let  $\mathfrak{G}(\Phi)$  be the set of all chains  $\{H_i\}_{i\geq 0}$  such that  $(G_\infty,G)$  and  $(H_\infty,G)$  are conjugate.

The set  $\mathfrak{G}$  is subdivided into classes of chains, such that  $(G_{\infty}, G, (eG_i))$  and  $(H_{\infty}, G, (eG_i))$  are pointed conjugate.

#### Type of action in terms of group chains

An equicontinuous Cantor minimal system  $(X, G, \Phi)$  is:

- regular if all group chains in  $\mathfrak{G}(\Phi)$  are equivalent;
- weakly regular if \$\mathcal{G}(\Phi)\$ contains a finite number of classes of equivalent group chains;
- irregular if \$\mathcal{G}(\Phi)\$ contains an infinite number of classes of equivalent group chains.

## The discriminant group (Dyer, Hurder, Lukina 2015)

Let  $\{G_i\}_{i\geq 0}$  be a group chain at a point x with partitions  $\{\mathcal{P}_i\}_{i\geq 0}$ . Let core  $G_i = C_i = \bigcap_{g\in G} gG_ig^{-1}$ , a normal subgroup of  $G_i$ . Then  $\{C_i\}_{i\geq 0}$  is a nested group chain, and

 $C_{\infty} = \lim_{\longleftarrow} \{G/C_i \to G/C_{i-1}\}$ 

is a profinite group, called the *limit core*.

The discriminant group is a subgroup

$$\mathcal{D}_x = \lim_{\longleftarrow} \{G_i/C_i \to G_{i-1}/C_{i-1}\} \subset C_{\infty}.$$

Note that  $G_{\infty} \cong C_{\infty}/\mathcal{D}_x$ .

# The discriminant function (Dyer, Hurder, Lukina 2015)

Each class of equivalent chains in  $\mathfrak{G}(\Phi)$  has a representative of the form  $\{g_iG_ig_i^{-1}\}_{i\geq 0}$ .

Since core  $g_i G_i g_i^{-1} = \text{core } G_i$ , there is a well-defined function

$$\mathcal{D}(\Phi): X \to C_{\infty}: x \mapsto \mathcal{D}_x,$$

called the discriminant function.

The limit core  $C_{\infty}$  is unique up to topological isomorphism, and  $\mathcal{D}(\Phi)$  is well-defined up to topological isomorphism of  $\mathcal{D}_x$  as a subset of  $C_{\infty}$ .

#### Question

How is the discriminant group related with  $Aut(X,G,\Phi)$  and with the degree of homogeneity of the action?

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# Properties of the discriminant function

### Theorem (Dyer, Hurder, Lukina 2015)

Let  $(X, G, \Phi)$  be an equicontinuous Cantor minimal system with a group chain  $\{G_i\}_{i\geq 0}$  at x, and let  $\mathcal{D}(\Phi)\colon X\to C_\infty$  be the discriminant function. Then:

- **9**  $\mathcal{D}(\Phi)$  is constant on the orbits of  $Aut(X, G, \Phi)$ .
- If y, z ∈ X are in the same orbit of (X, G, Φ), then the discriminant groups D<sub>y</sub> and D<sub>z</sub> are isomorphic as topological groups.
- If the action  $(X, G, \Phi)$  is regular or weakly regular, then for every  $y \in X$  the discriminant groups  $\mathcal{D}_x$  and  $\mathcal{D}_y$  are isomorphic as topological groups.

# Discriminant group for weakly regular actions

#### Theorem (Dyer, Hurder, Lukina 2015)

Let  $(X, G, \Phi)$  be an equicontinuous Cantor minimal system with a group chain  $\{G_i\}_{i\geq 0}$  at x, and let  $\mathcal{D}(\Phi)\colon X\to C_\infty$  be the discriminant function. Then:

- **(**) The action  $(X, G, \Phi)$  is regular if and only if  $\mathcal{D}(\Phi)$  is a trivial map.
- If for some x ∈ X the discriminant group D<sub>x</sub> is finite, then the action (X, G, Φ) is weakly regular.

If an action  $(X,G,\Phi)$  is weakly regular, then the discriminant group is finite or infinite.

If  $(X, G, \Phi)$  is weakly regular, and  $\mathcal{D}_x$  is finite, then the action  $(X, G, \Phi)$  is *tame*.

### Weakly regular action with finite discriminant group

Let  ${\Gamma_i}_{i\geq 0}$  be a chain of normal subgroups in a group  $\Gamma$ .

Let H be a finite simple group, and let  $K \subset H$  be a non-trivial subgroup. Since H is simple, K is not normal in H.

Let  $G = H \times \Gamma$ , and  $G_i = K \times \Gamma_i$ ,  $i \ge 0$ .

Then  $G_i$  is a normal subgroup of  $G_0 = K \times \Gamma$ , and  $(G_{\infty}, G)$  is weakly regular.

Also,  $G/G_i = (H \times \Gamma)/(K \times \Gamma_i) = H/K \times \Gamma/\Gamma_i$  and  $G_{\infty} = H/K \times \Gamma_{\infty}$ .

We have

$$C_i = \bigcap_{g \in G} gG_i g^{-1} = \{e\} \times \Gamma_i.$$

Then  $G_i/C_i = (K \times \Gamma_i)/\Gamma_i$ , and  $\mathcal{D}_x \cong K$ .

## Weakly regular action with infinite discriminant group

Let  $\Gamma = \mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\}$ , and p, q be distinct primes. Define  $\Gamma_i = \{(ap^i, bq^i) \mid a, b, \in \mathbb{Z}\}, \text{ and } \Gamma_i^T = \{(aq^i, bp^i) \mid a, b, \in \mathbb{Z}\}.$ Let  $H = \mathbb{Z}_2 = \{1, t\}$ , and  $\theta \colon H \to Aut(\mathbb{Z}^2)$  be given by  $\theta(t)(a,b) = (b,a).$ Let  $G = \Gamma \rtimes H$ , and  $G_i = \Gamma_i \times \{1\}$ . Since  $gG_ig^{-1} = G_i^T$  then  $C_i = core \ G_i = G_i \cap (\Gamma_i^T \times \{1\}) = \{(ap^i q^i, bp^i q^i, 1) \mid a, b \in \mathbb{Z}\}.$ We have  $card(G_i/C_i) = p^i q^i$ , and the maps

$$G_i/C_i \to G_{i-1}/C_{i-1}$$

are surjective. Then  $\mathcal{D}_x$  is infinite.

Equivalences

### Further questions

### Open problems

- Let (X,G) and (Y,G) be orbit equivalent minimal Cantor actions. What happens with the discriminant function D(Φ) of the actions under the map h?
- Let (X, G) be a minimal Cantor action associated to the action of a self-similar group G. How is the discriminant function  $\mathcal{D}(\Phi)$  of the action (X, G) related to the self-similar structure of the action, and the geometry of the space X?
- **③** Is there a relationship between the discriminant function  $\mathcal{D}(\Phi)$  and the full group of an action  $(X, G, \Phi)$ ?

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