Groupoid Methods in Free Analysis

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Matrix Bundles

- Let M be a complex manifold and let $\mathfrak{P}$ be a holomorphic principal $G := PGL(n, \mathbb{C})$-bundle on M. Let $\mathfrak{P}[M_n(\mathbb{C})]$ be the associated holomorphic $M_n(\mathbb{C})$ bundle over M.
- Let $\Gamma_h(M, \mathfrak{P}[M_n(\mathbb{C})])$ be the algebra of holomorphic sections of $\mathfrak{P}[M_n(\mathbb{C})]$ and let $\Gamma_c(M, \mathfrak{P}[M_n(\mathbb{C})])$ be the algebra of continuous sections of $\mathfrak{P}[M_n(\mathbb{C})]$.
- $\Gamma_h(M, \mathfrak{P}[M_n(\mathbb{C})])$ and $\Gamma_c(M, \mathfrak{P}[M_n(\mathbb{C})])$ are both topological algebras in the compact open topology.

Free Complex Analysis vs Classical Complex Analysis

- Free Complex Analysis must be studied in spaces like $\Gamma_h(M, \mathfrak{P}[M_n(\mathbb{C})])$.
- Classical Complex Analysis is the study of holomorphic cross sections of trivial holomorphic line bundles.
- More accurately: Complex analysis begins with polynomial functions, which are cross sections of trivial holomorphic line bundles, but free complex analysis begins with algebras of generic matrices. These live only in certain nontrivial matrix bundles.

The Problem

- Let X be a compact subset of M. How to put a norm on $\Gamma_c(X, \mathfrak{P}[M_n(\mathbb{C})])$ and $\Gamma_h(X, \mathfrak{P}[M_n(\mathbb{C})])$ so that $\Gamma_c(X, \mathfrak{P}[M_n(\mathbb{C})])$ is a C^*-algebra? The transition functions that define $\mathfrak{P}$ take values in $G = PGL(n, \mathbb{C})$ not in $K = PU(n, \mathbb{C})$.
- Recall [Tomiyama & Takesaki, 1961] and [Fell, 1961] There is a categorical equivalence between iso-classes of unital n-homogeneous C^*-algebras with compact spectrum X and equivalence classes of matrix bundles $\mathfrak{Q}[M_n(\mathbb{C})]$, where $\mathfrak{Q}$ is a principal $K = PU(n, \mathbb{C})$ bundle on X.
- Goal: To study the closure of $\Gamma_h(X, \mathfrak{P}[M_n(\mathbb{C})])$ in $\Gamma_c(X, \mathfrak{P}[M_n(\mathbb{C})])$ using [Arveson, 1969].

Mixed News

The Good

- Because $PGL(n, \mathbb{C})/PU(n, \mathbb{C})$ is contractible (polar decomposition), a $PGL(n, \mathbb{C})$-principal bundle $\mathfrak{P}$ can always be reduced to a principal $PU(n, \mathbb{C})$-bundle $\mathfrak{Q}$ defined over the same base M.
- This means that there is a K-equivariant homeomorphism from the bundle space of $\mathfrak{Q}$ into that of $\mathfrak{P}$ whose image projects down onto the base space of $\mathfrak{P}$.
- The bundles $\mathfrak{P}[M_n(\mathbb{C})]$ and $\mathfrak{Q}[M_n(\mathbb{C})]$ are isomorphic as topological bundles.
- $\Gamma_c(X, \mathfrak{P}[M_n(\mathbb{C})]) \simeq \Gamma_c(X, \mathfrak{Q}[M_n(\mathbb{C})])$ and $\Gamma_c(X, \mathfrak{Q}[M_n(\mathbb{C})])$ is a C^*-algebra.
- If $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are reductions of $\mathfrak{P}$, then $\Gamma_c(X, \mathfrak{Q}_1[M_n(\mathbb{C})]) \simeq \Gamma_c(X, \mathfrak{Q}_2[M_n(\mathbb{C})])$ as C^*-algebras.

Mixed News
The Bad

- The bundle $\mathfrak{Q}[M_n(\mathbb{C})]$ has no holomorphic structure, in general, and it makes no sense to speak of elements of the image of $\Gamma_h(X, \mathfrak{P}[M_n(\mathbb{C})])$ in $\Gamma_c(X, \mathfrak{Q}[M_n(\mathbb{C})])$ as consisting of holomorphic sections.
- Further, a C^*-isomorphism between $\Gamma_c(X, \mathfrak{Q}_1[M_n(\mathbb{C})])$ and $\Gamma_c(X, \mathfrak{Q}_2[M_n(\mathbb{C})])$ need not carry the image of $\Gamma_h(X, \mathfrak{P}[M_n(\mathbb{C})])$ in the first to the image in the second.
- Further, still: The maximum modulus theorem and other aspects of complex analysis fail to hold.

Nevertheless...

- Let $\overline{\mathcal{D}}$ be a Stein-compact domain in M
- $\partial\mathcal{D}$ - the Shilov boundary of $\mathcal{D}$.
- $\partial_e\mathcal{D}$ - the Choquet boundary.
- Let $A(\partial\mathcal{D})$ be the closure of the image of $\Gamma_h(\overline{\mathcal{D}}, \mathfrak{P}[M_n(\mathbb{C})])$ in $\Gamma_c(\partial\mathcal{D}, \mathfrak{Q}[M_n(\mathbb{C})])$.

Theorem (Griesenauer)

Each point evaluation from $\partial_e\mathcal{D}$ is a boundary representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{Q}[M_n(\mathbb{C})])$ for $A(\partial\mathcal{D})$.

Corollary

$\Gamma_c(\partial\mathcal{D}, \mathfrak{Q}[M_n(\mathbb{C})])$ is the C^*-envelope of $A(\partial\mathcal{D})$.

Free Polynomial Functions

- For $\mathfrak{z} := (Z_1, Z_2, \cdots, Z_d) \in M_n(\mathbb{C})^d$ and $f \in \mathbb{C}\langle X_1, X_2, \cdots X_d\rangle$, let $f(\mathfrak{z}) = \sum_w a_w Z^w$, where $f = \sum_w a_w X^w$, where w is a word in the free semigroup on d letters, and where $Z^w = Z_{i_1} Z_{i_2} \cdots Z_{i_n}$, $X^w = X_{i_1} X_{i_2} \cdots X_{i_n}$, and $w = i_1 i_2 \cdots i_n$
- The algebra of all such functions on $M_n(\mathbb{C})^d$ is called the algebra of d-generic $n \times n$ matrices and is denoted $\mathbb{G}(d, n)$.
- The algebra of all polynomial functions $p : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$ such that $p(s^{-1}\mathfrak{z}s) = p(\mathfrak{z})$ (the invariant polynomials) is denoted $\mathbb{I}(d, n)$
- The algebra generated by $\mathbb{G}(d, n)$ and $\mathbb{I}(d, n)$ is called the trace algebra and is denoted $\mathbb{S}(d, n)$.

Free Holomorphic Functions

Free Entire Functions, really

Theorem

The closure of $\mathbb{S}(d, n)$ in the topology of uniform convergence on compact sets on the space $C(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ is the set of all holomorphic functions $f : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$ such that $f(s^{-1}\mathfrak{z}s) = s^{-1}f(\mathfrak{z})s$ for all $s \in GL(n, \mathbb{C})$ and all $\mathfrak{z} \in M_n(\mathbb{C})^d$.

Notation

We write $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ for the space of entire functions such that $f(s^{-1}\mathfrak{z}s) = s^{-1}f(\mathfrak{z})s$.

- $\mathcal{V}(d,n)$ - all $\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d)$ such that the Z_i generate $M_n(\mathbb{C})$ as an algebra.
- $\mathcal{V}(d,n)$ is open, G-invariant and Zariski dense in $M_n(\mathbb{C})^d$.
- G acts freely and properly on $\mathcal{V}(d,n)$ and so $\mathcal{V}(d,n)$ is the bundle space of a principal G-bundle over $\mathcal{V}(d,n)/G$.

Theorem

[Procesi, 1974] $\mathcal{V}(d,n)/G$ is biholomorphically isomorphic to an open set, $Q_0(d,n)$, in the smooth points of Spec$(\mathbb{I}(d,n))$

- $\mathfrak{M}(d,n)$ - the associated $M_n(\mathbb{C})$-bundle.

Where Free Analytic Functions Live

Theorem (GMS)

For all (n, d), $Q_0(d, n)$ parametrizes all the similarity classes of algebraically irreducible n-dimensional representations of $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ and for $(n, d) \neq (2, 2)$, $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G \simeq \Gamma_h(Q_0(d, n), \mathfrak{M}(d, n))$.

- Thus free holomorphic functions on $M_n(\mathbb{C})^d$ live as sections of a natural holomorphic matrix bundle erected over the similarity classes of irreducible n-dimensional representations of the free algebra on d generators. Further, the bundle is non-trivial since $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ is a (noncommutative) integral domain.

An Alternate Approach?

Inspired by the groupoid of a cover.

- Another way to embed $\Gamma_h(X, \mathfrak{P}[M_n(\mathbb{C})]$ in homogeneous C^*-algebra.
- Fix an open cover $\mathcal{U}$ of M that trivializes $\mathfrak{P}$. Let $\{g_{UV}\}_{U,V\in\mathcal{U}}$ be a cocycle that gives $\mathfrak{P}$.
- Fix isomorphisms F_U from $\mathfrak{P}[M_n(\mathbb{C})]|_U$ to the product bundle "$U \times M_n(\mathbb{C})$ over U".
- Realize $\Gamma_h(X, \mathfrak{P}[M_n(\mathbb{C})]$ as the subalgebra of $C_b(\coprod_{U\in\mathcal{U}} U, M_n(\mathbb{C}))$ consisting of $\{f_U\}_{U\in\mathcal{U}}$ such that $f_U(x) = g_{U,V}(x)(f_V(x))$.

Further Reading I

- Arveson, W. B. (1969). Subalgebras of C^*-algebras. *Acta Math.*, 123, 141–224.
- Fell, J. M. G. (1961). The structure of algebras of operator fields. *Acta Math.*, 106, 233–280.
- Procesi, C. (1974). Finite dimensional representations of algebras. *Israel J. Math.*, 19, 169–182.
- Tomiyama, J. & Takesaki, M. (1961). Applications of fibre bundles to the certain class of C^*-algebras. *Tôhoku Math. J. (2)*, 13, 498–522.