Wavelets for higher-rank graph C^*-algebras

Judith Packer (U. Colorado, Boulder)
with C. Farsi (U. Colorado, Boulder), E. Gillaspy (U. Colorado, Boulder) and S. Kang (U. Otago)

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Background on graph C^*-algebras:

Higher rank graph C^*-algebras were introduced by A. Kumjian and D. Pask in 2000. These are generalizations of algebras associated to directed graphs, which in turn generalized the Cuntz algebras $\mathcal{O}_n$ generated by families of isometries first studied by J. Cuntz in 1977.

Definition: (Kumjian-Pask, 2000) Let $k \geq 1$. A k-graph, or higher-rank graph, is a countable small category Λ equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorization property: if $d(\lambda) = m + n$, then there exist unique μ and ν such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

Background on graph C^*-algebras, continued:

Examples

1. For $k = 1$, Λ is the path category of a connected graph, and d is the length function.
2. For $k > 1$, Λ is a collection of paths in a multi-colored graph; here paths have a "degree" or "shape" rather than a length.
3. Two paths μ and ν can be concatenated if $s(\mu) = r(\nu)$, and then $d(\mu\nu) = d(\mu) + d(\nu)$.

By the work of Kumjian and Pask [KP], C^*-algebras can be associated to higher-rank graphs and higher-rank graphs equipped with a 2-cocycle. Also Kumjian and Pask showed that a k-graph Λ has associated to it a path groupoid $\mathcal{G}_\Lambda$.

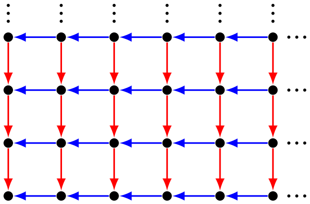
Fundamental example, k-graph

Let

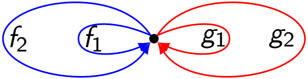
$$\Omega_k = \{(m,n) : m,n \in \mathbb{N}^k : m \leq n\}.$$

The structure maps are given by $r(m,n) = m, s(m,n) = n$, so that $(\ell, n) = (\ell, m)(m,n)$ and then $d(m,n) = n - m$ is a functor.

Ω_2 can be represented by the following bi-colored graph with the factorization rules.



Another example



We choose the following factorization rules:

$$f_1 g_1 = g_1 f_1 \text{ and } f_1 g_2 = g_2 f_1$$
$$f_2 g_1 = g_1 f_2 \text{ and } f_2 g_2 = g_2 f_2$$

Notation and terminology associated to k-graphs

- $\Lambda^n := d^{-1}(n)$ denotes the set of paths of degree n.
- The paths Λ^0 of degree 0 are called vertices.
- For $v, w \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$v\Lambda^n w = \{\lambda \in \Lambda^n : r(\lambda) = v,\ s(\lambda) = w\}$$

- A k-graph is called finite if Λ^n is finite for all $n \in \mathbb{N}^k$.
- We say that Λ has no sources if $v\Lambda^{e_i} \neq \emptyset$ for all $v \in \Lambda^0$ and $i \in \{1, \ldots, k\}$..

Notation for k-graphs, continued:

- We say that a k-graph Λ is strongly connected if, for all $v, w \in \Lambda^0$, $v\Lambda w \neq \emptyset$.
- For $1 \leq i \leq k$, let A_i be the matrix of $M_{\Lambda^0}(\mathbb{N})$ with entries $A_i(v, w) = |v\Lambda^{e_i} w|$, the number of paths from w to v with degree e_i; we call the A_i the vertex adjacency matrices of Λ.

The higher-rank graph algebra $C^*(\Lambda)$

Definition 1

Let Λ be a finite k-graph with no sources. A Cuntz-Krieger Λ-family consists of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ such that

(CK1) $\{t_v : v \in \Lambda^0\}$ are mutually orthogonal projections;

(CK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$;

(CK4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$.

The C^*-algebra $C^*(\Lambda)$ of Λ is generated by a universal Λ-family $\{s_\lambda\}$. We write $p_v := s_v$ for $v \in \Lambda^0$.

Representations of graph algebras

There is a very well-developed theory of representations of Cuntz-Krieger C^*-algebras associated to directed graphs (where $k = 1$) on Hilbert spaces. To list just a few names, O. Bratteli and P. Jorgensen, K. Kawamura, D. Dutkay and P. Jorgensen, and M. Marcolli and A. Paolucci have studied representations of Cuntz-Krieger algebras on a variety of Hilbert spaces. In particular, in these papers the Perron-Frobenius theory has been used on the incidence matrices involved to construct measures giving interesting Hilbert spaces on which to represent these algebras.

In addition, M. Marcolli and A. Paolucci [MP] represented Cuntz-Krieger algebras on Hilbert spaces associated to fractals, extending results of A. Jonnson [Jo] to construct a wavelet family associated to particular representations of these C^*-algebras.

The Infinite Path Space for a k-graph Λ

In 2015, A. an Huef, M. Laca, I. Raeburn, and A. Sims [aHLRS2] used the Perron-Frobenius theory associated to incidence matrices for finite higher-rank graphs to construct a probability measure M on the so-called infinite path space Λ^∞ associated to Λ, and used this measure in the study of the abelian core of $C^*(\Lambda)$ as well as to construct KMS states associated to the graph C^*-algebras.

Farsi, Gillaspy, Kang, and P. represented $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$, so as to generalize the construction of "wavelets".

The Infinite Path Space of Λ, continued

Recall the basic example of a k-graph:

$$\Omega_k := \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}.$$

Recall that with composition defined by $(p,q)(q,m) = (p,m)$ and degree map given by $d(p,q) = q - p$, Ω_k is a k-graph with no sources.

Definition 2

Let Λ be a finite k-graph with no sources. An infinite path in a k-graph Λ is a k-graph morphism $x : \Omega_k \to \Lambda$. We write Λ^∞ for the collection of all infinite paths and call it the infinite path space of Λ. For each $p \in \mathbb{N}^k$, we define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m,n) = x(m+p, n+p)$ for $x \in \Lambda^\infty$. For $\lambda \in \Lambda$ we define $Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}$ and we call it a *cylinder set*. The cylinder sets $\{Z(\lambda)\}$ are a basis for the topology on Λ^∞. Note that since Λ is finite, Λ^∞ is compact.

The Perron-Frobenius Measure on Λ^∞

Let Λ be a strongly connected k-graph with vertex matrices $A_1, \ldots, A_k$. Because Λ is strongly connected, these matrices will be irreducible and nonnegative. Because of the factorization property of k-graphs, these matrices will commute.

The Perron-Frobenius theory shows that there is a unique common nonnegative unimodular Perron-Frobenius eigenvector x^Λ for the vertex matrices A_i. We will call x^Λ the Perron-Frobenius eigenvector of the k-graph Λ.

Eigenvalues associated to x^Λ are called the Perron-Frobenius eigenvalues. The Perron-Frobenius eigenvalue for A_i is the spectral radius $\rho(A_i)$ of A_i.

The Measure on Λ^∞, continued

The following definition was originally given in Proposition 8.1 of [aHLRS2].

Definition 3

Let Λ be a strongly connected finite k-graph with the vertex matrices A_i. Define a measure M on cylinder sets of Λ^∞ by

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)} \quad \text{for all } \lambda \in \Lambda, \tag{1}$$

where $\rho(\Lambda) = (\rho(A_1), \ldots, \rho(A_k))$ and x^Λ is the unimodular Perron-Frobenius eigenvector of Λ described in the previous slide. We call M the Perron-Frobenius measure on the infinite path space Λ^∞.

The representation of $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$

To construct the representation of the graph algebra $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$ we need to construct partial isometries satisfying the CK conditions.

More on the infinite path space

Notation: Let Λ be a strongly connected finite k-graph. Define for each $\lambda \in \Lambda$ a "prefixing map" $\sigma_\lambda : Z(s(\lambda)) \subset \Lambda^\infty \to Z(\lambda) \subset \Lambda^\infty$ by $\sigma_\lambda(x) = \lambda x$, and define "coding maps" $\{\sigma^m : \Lambda^\infty \to \Lambda^\infty\}_{m \in \mathbb{N}^k}$ by $\sigma^m(x) = x(m, \infty)$. (Note the "prefixing maps" insert a "shape", and the "coding maps" delete a "shape" from a path.)

Then

(a) For each $m \in \mathbb{N}^k$, the family $\{\sigma_\lambda : d(\lambda) = m\}$ and σ_m satisfy

$$\sigma^m(\sigma_\lambda(x)) \;=\; x, \; x \in Z(\lambda).$$

(b) If $v \in \Lambda^0$, then $\sigma_v = id$, and $M(Z(v)) > 0$.

(c) Let $R_\lambda = \sigma_\lambda(Z(\lambda))$. For each $\lambda \in \Lambda, \nu \in s(\lambda)\Lambda$, we have $R_\nu \subseteq Z(\lambda)$ (up to a set of measure 0), and $\sigma_\lambda \sigma_\nu = \sigma_{\lambda\nu}$ a.e.

(d) The coding maps satisfy $\sigma^m \circ \sigma^n = \sigma^{m+n}$ for any $m, n \in \mathbb{N}^k$.

The representation of $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$:

We say that a k-graph Λ is aperiodic if for each $v \in \Lambda^0$, there exists $x \in Z(v)$ such that for all $m \neq n \in \mathbb{N}^k$, we have $\sigma^m(x) \neq \sigma^n(x)$.

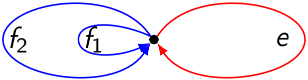
Theorem 4 (FGKP)

Let Λ be a finite k-graph with no sources, and let the prefixing maps $\{\sigma_\lambda : Z(s(\lambda)) \subset \Lambda^\infty \to Z(\lambda) \subset \Lambda^\infty\}$ and coding maps $\{\sigma^m : \Lambda^\infty \to \Lambda^\infty\}_{m \in \mathbb{N}^k}$ be defined on the measure space (Λ^∞, M) as previously. For each $\lambda \in \Lambda$, define $S_\lambda \in B(L^2(\Lambda^\infty, M))$ by

$$S_\lambda \xi(x) = \chi_{Z(\lambda)}(x)\rho(\Lambda)^{d(\lambda)/2}\xi(\sigma^{d(\lambda)}(x)).$$

Then the operators $\{S_\lambda\}_{\lambda \in \Lambda}$ generate a representation of $C^*(\Lambda)$. If Λ is aperiodic then this representation is faithful.

Example of this representation



In this example, we set our factorization rules to be:

$$f_2 e = e f_1 \text{ and } f_1 e = e f_2$$

By these factorization rules, we see that if a particular infinite path in $x \in \Lambda^\infty$ has infinitely many of both $\{e\}$ and $\{f_i\}_{i=1}^2$, it can be chosen to be of the form

$$e f_{i_1} e f_{i_2} e f_{i_3} \cdots .$$

The two incidence matrices corresponding to the two colorad of edges and one vertex are 1×1 and we have $(A_1) = (1)$, $(A_2) = (2)$. Therefore the Perron Frobenius-measure on cylinder sets is:

$$M(Z(e)) = 1,\ M(Z(ef_i)) = 1/2,\ i = 1,\ 2,\ \text{etc.}$$

Representation example, continued

Using the Theorem, we will find isometries S_e, S_{f_1}, and S_{f_2} on $L^2(\Lambda^\infty, M)$ satisfying

$$S_e^* S_e \ = \ S_{f_1}^* S_{f_1} \ = \ S_{f_2}^* S_{f_2} = \mathrm{Id},$$

$$S_e S_e^* = S_{f_1} S_{f_1}^* + S_{f_2} S_{f_2}^* = \mathrm{Id}.$$

and finally

$$S_e S_{f_1} \ = \ S_{f_2} S_e, \ S_e S_{f_2} = S_{f_1} S_e.$$

So, for $\xi \in L^2(\Lambda^\infty, M)$ and $x \equiv ef_{i_1} ef_{i_2} ef_{i_3} \cdots$,

Representation example, conclusion

we calculate

$$S_e(\xi)(x) = \chi_{Z(e)}(x)1^{1/2}2^{0/2}\xi(\sigma^{(1,0)}x) \;=\; \xi(ef_{i_1+1}ef_{i_2+1}ef_{i_3+1}\cdots);$$

where the addition in the subscripts of f is taken modulo 2, and

$$S_{f_1}(\xi)(x) = \;2^{1/2}\chi_{Z(f_1)}(x)\xi(\sigma^{(0,1)}x) = \;2^{1/2}\chi_{Z(f_1)}(x)\xi(ef_{i_2+1}ef_{i_3+1}\cdots);$$

$$S_{f_2}(\xi)(x) = \chi_{Z(f_2)}(x)1^{0/2}2^{1/2}\xi(\sigma^{(0,1)}x) = 2^{1/2}\chi_{Z(f_2)}(x)\xi(ef_{i_2+1}ef_{i_3+1}\cdots)$$

and verify that the appropriate CK relations are satisfied.

Wavelets associated to the representations of $C^*(\Lambda)$

We now follow generalize a construction of M. Marcolli and A. Paolucci for Cuntz-Krieger C^*-algebras (these are C^*-algebras associated to directed graphs, $k = 1$.) Their work in turn was motivated by work of A. Jonnson. In the process, we construct an orthogonal decomposition of $L^2(\Lambda^\infty, M)$, which we call a wavelet decomposition. We have extended the construction to obtain wavelets corresponding to an arbitrary positive degree, thus answering a question posed by Aidan Sims.

Wavelets associated to the representations of $C^*(\Lambda)$, continued

To construct orthogonal subspaces of $L^2(\Lambda^\infty, M)$, we first set the initial space $\mathcal{V}_{0,\lambda}$ equal to the subspace spanned by the functions $\{\Theta_v : v \in \Lambda^0\}$. Note that the functions $\{\Theta_v : v \in \Lambda^0\}$ form an orthogonal set in $L^2(\Lambda^\infty, M)$, whose span includes the constant functions on Λ^∞.

To construct the wavelet subspace $\mathcal{W}_{0,\Lambda}$, fix $v \in \Lambda^0$, and fix $(j_1, j_2, \cdots, j_k) \in [\mathbb{N}^+]^k$. Let

$$D_v = \{\lambda : d(\lambda) = (j_1, \ldots, j_k) \text{ and } r(\lambda) = v\},$$

and write d_v for $|D_v|$ (since Λ is a finite k-graph, $d_v < \infty$).

Wavelets associated to the representations of $C^*(\Lambda)$, continued

Define an inner product on $\mathbb{C}^{d_v}$ by

$$\langle \vec{v}, \vec{w} \rangle = \sum_{\lambda \in D_v} \overline{v_\lambda} w_\lambda \rho(\Lambda)^{(-j_1,\ldots,-j_k)} x^{\Lambda}_{s(\lambda)} \tag{2}$$

and let $\{c^{m,v}\}_{m=1}^{d_v-1}$ be an orthonormal basis for the orthogonal complement of $(1,\ldots,1) \in \mathbb{C}^{D_v}$ with respect to this inner product.

For each pair (m,v) with $1 \leq m \leq d_v - 1$ and v a vertex in Λ^0, define

$$f^{m,v} = \sum_{\lambda \in D_v} c^{m,v}_\lambda \Theta_\lambda.$$

Wavelets associated to the representations of $C^*(\Lambda)$, continued

By the definition of M on Λ^∞, since the vectors $c^{m,v}$ are orthogonal to $(1, \cdots, 1)$ in the inner product (2), we have

$$\int_{\Lambda^\infty} f^{m,v}\, dM \;=\; 0.$$

We also easily check

$$\int_{\Lambda^\infty} f^{m,v}\overline{f^{m',v'}}\, dM = \delta_{v,v'}\delta_{m,m'},$$

so that $\{f^{m,v}\}$ are an orthonormal set in $L^2(\Lambda^\infty, M)$.

Following Marcolli and Paolucci, we define the wavelet subspace

$$\mathcal{W}_{0,\Lambda} := \overline{\text{span}}\{f^{m,v} : v \in \Lambda^0, 1 \le m \le d_v - 1\}.$$

Wavelets associated to the representations of $C^*(\Lambda)$, main theorem

Theorem 5 (FGKP)

Let Λ be a strongly connected finite k-graph. For each fixed $j \in \mathbb{N}_+$ and $v \in \Lambda^0$, let $C_{j,v} := \{\lambda \in \Lambda : s(\lambda) = v, d(\lambda) = (j \cdot j_1, j \cdot j_2, \ldots, j \cdot j_k)\}$, and let S_λ be the operator on $L^2(\Lambda^\infty, M)$ described in Theorem 4. Then $\{S_\lambda f^{m,v} : v \in \Lambda^0,\ \lambda \in C_{j,v},\ 1 \leq m \leq d_v - 1\}$ is an orthonormal set, so that for $\lambda \in C_{j,v},\ \mu \in C_{i,v'}$ with $0 < i < j$, we have

$$\int_{\Lambda^\infty} S_\lambda f^{m,v} \overline{S_\mu f^{m',v'}}\, dM = 0\ \forall\ m, m'.$$

Thus, defining $\mathcal{W}_{j,\Lambda} := \overline{span}\{S_\lambda f^{m,v} : v \in \Lambda^0, \lambda \in C_{j,v}, 1 \leq m \leq d_v - 1\}$ for $j \geq 1$, we obtain an orthogonal decomposition

$$L^2(\Lambda^\infty, M) = \mathcal{V}_0 \oplus \bigoplus_{j=0}^\infty \mathcal{W}_{j,\Lambda}.$$

Example of wavelet decomposition for 2-graph algebra:

Example 6

We return to the case where Λ has one vertex and three edges (two blue f_1 and f_2 and one red, e).
Let M be the Perron-Frobenius measure on Λ^∞, let ϕ be the constant function 1 on Λ^∞, let $(j_1, j_2) = (1, 1)$, and let

$$\psi = \chi_{Z(ef_1)} - \chi_{Z(ef_2)}.$$

By using the theorem or direct calculation we verify that

$$\{\phi\} \cup \cup_{j=0}^{\infty} \{S_\lambda(\psi) : \lambda \in \Lambda, d(\lambda) = (j, j)\}$$

is an orthonormal basis for $L^2(\Lambda^\infty)$.

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