Mapping Groupoids for Orbispaces

Dorette Pronk[1] Laura Scull[2]

[1]Dalhousie University, Halifax, NS

[2]Fort Lewis College, Durango, CO

AMS Fall Southeastern Sectional Meeting
Memphis, October 17, 2015

Outline

This talk is based on:

- Vesta Coufal, Dorette Pronk, Carmen Rovi, Laura Scull, Courtney Thatcher, Orbispaces and their mapping spaces via groupoids: a categorical approach, *Contemporary Mathematics* **641** (2015), pp. 135–166
- Dorette Pronk, Laura Scull, Orbifold mapping spaces as pseudo colimits, in progress

Topological Groupoids

- Let **Top** be a category of 'nice' topological spaces. ('Nice' means: compactly generated, locally compact, Hausdorff, paracompact, generalized topological manifold.)
- A **topological groupoid** is an internal groupoid in **Top**,

$$\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1 \xrightarrow{\mathrm{inv}} \mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0 \quad (u : \mathcal{G}_0 \to \mathcal{G}_1)$$

Etale Groupoids

- A topological groupoid $\mathcal{G}$ is called **étale** when its structure maps are local homeomorphisms.
- For any $x \in \mathcal{G}_0$, its **isotropy group** $\mathcal{G}_x$ is defined as $(s,t)^{-1}(x,x) = s^{-1}(x) \cap t^{-1}(x) \subseteq \mathcal{G}_1$.
- When the groupoid is étale all isotropy groups are discrete.

Proper Groupoids

- A topological groupoid $\mathcal{G}$ is called **proper** when the diagonal,

$$(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0,$$

  is a proper map (i.e., closed with compact fibers).
- The isotropy groups in a proper groupoid are all compact.

Orbigroupoids

Definition

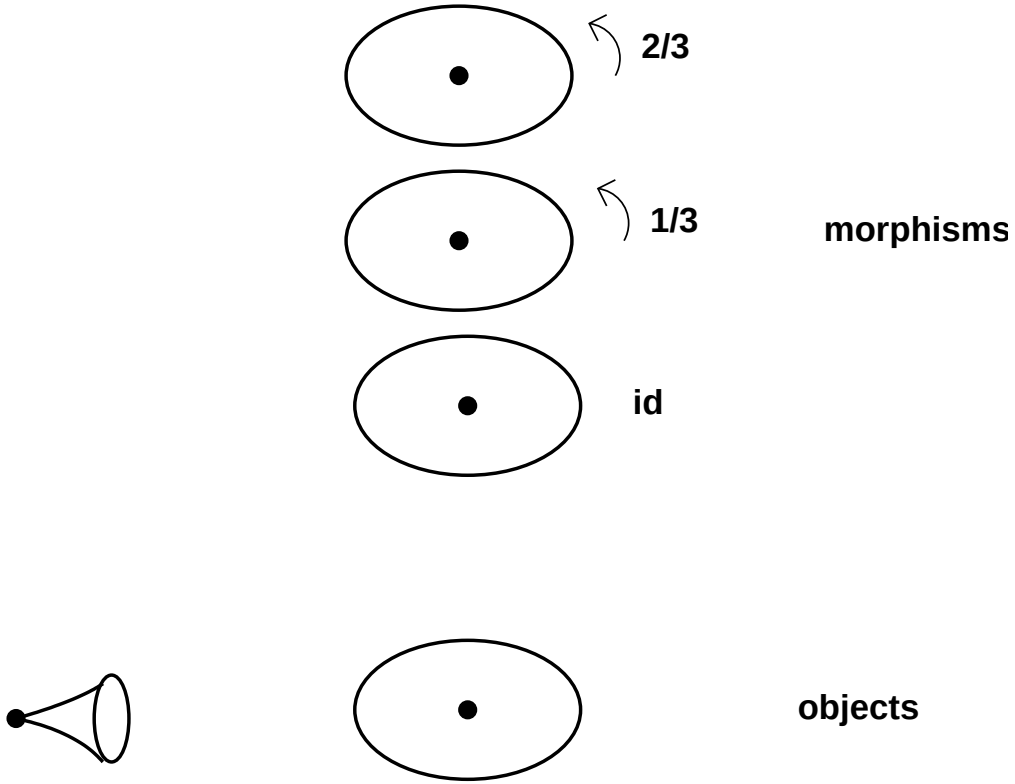
1. A topological groupoid is an **orbigroupoid** if it is both étale and proper.
2. The quotient space,

$$\mathcal{G}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}_0 \twoheadrightarrow X_{\mathcal{G}}$$

is also called the underlying space of the orbispace.

Examples

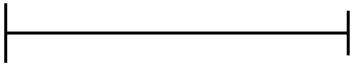
A Cone of Order 3



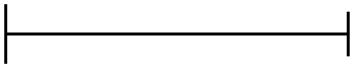
This is a translation groupoid, $\mathbb{Z}/3 \ltimes D$.

Examples

The Unit Interval



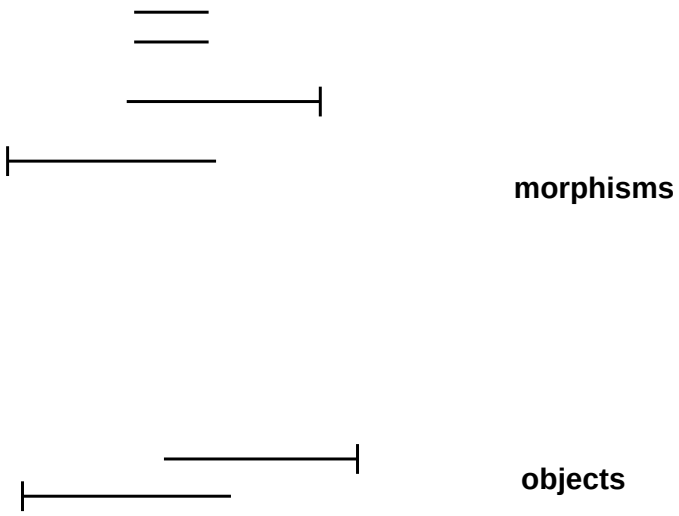
morphisms



objects

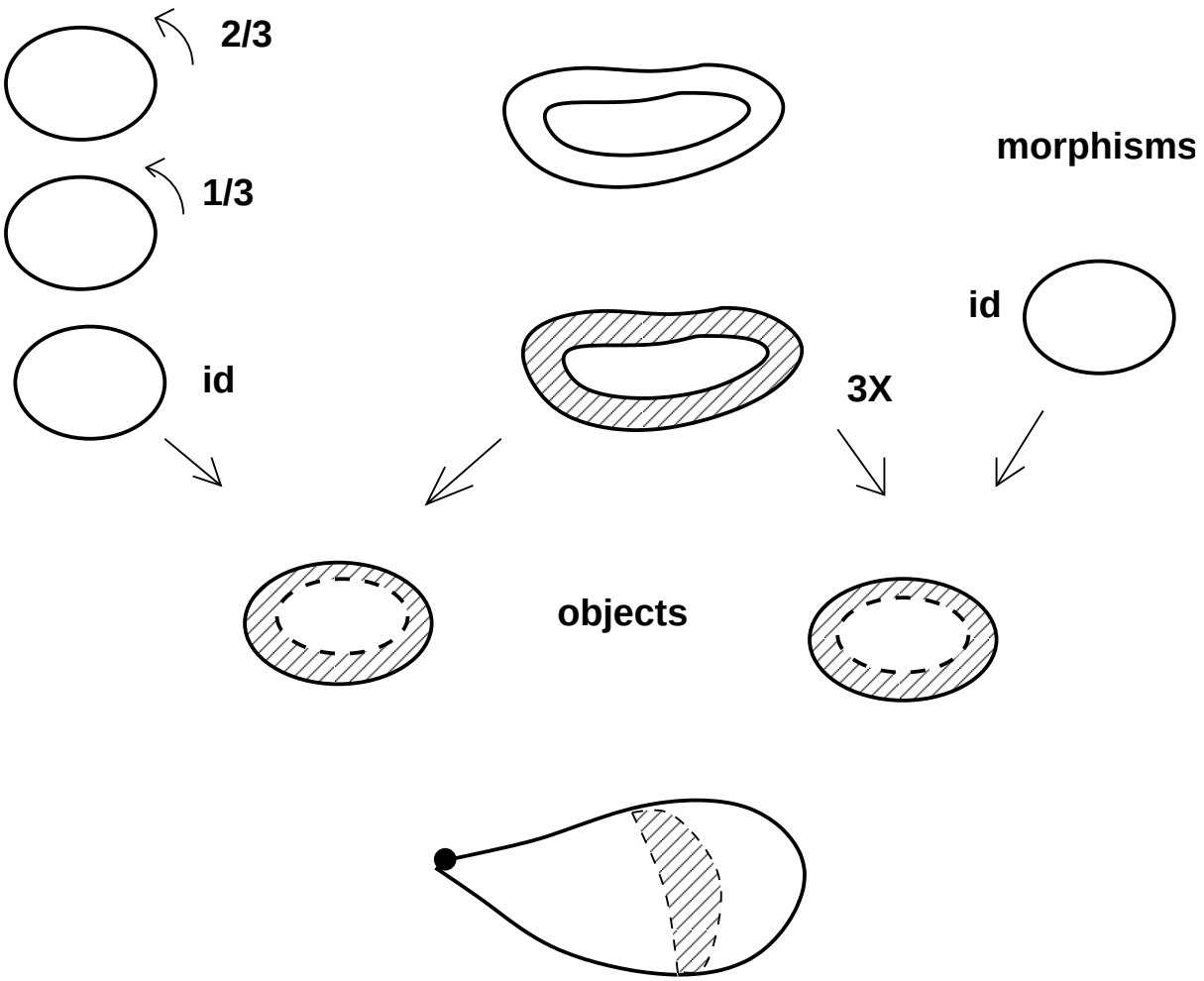
Examples

The Unit Interval again

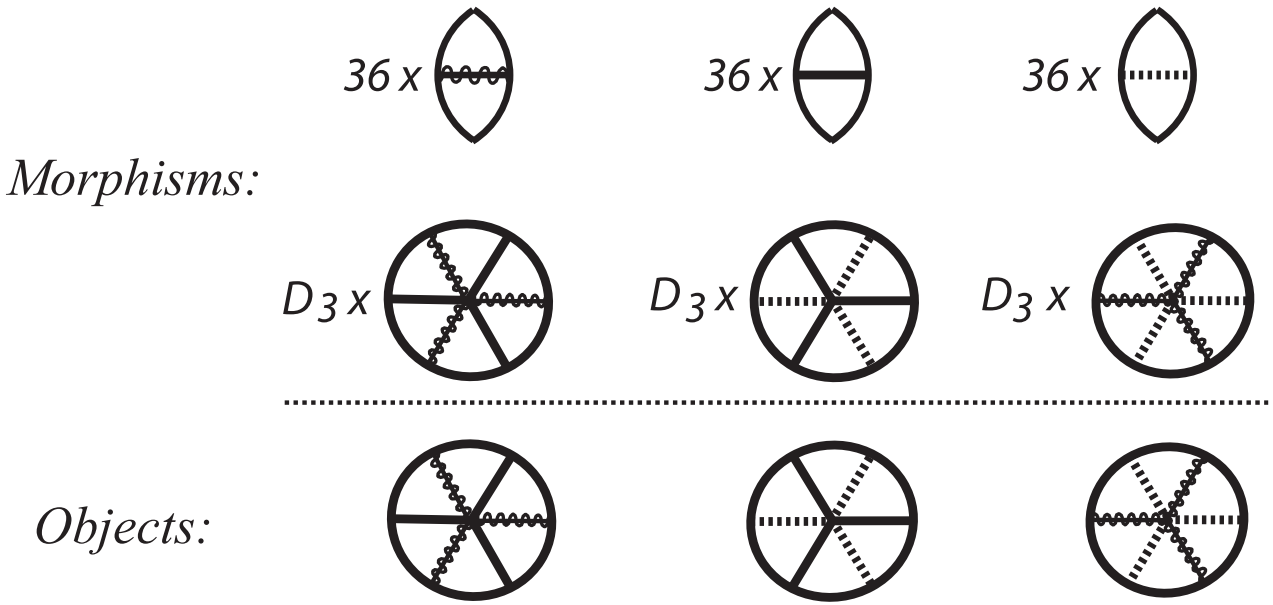


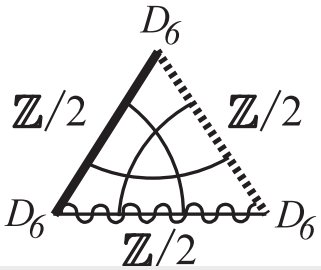
Examples

The Teardrop Groupoid



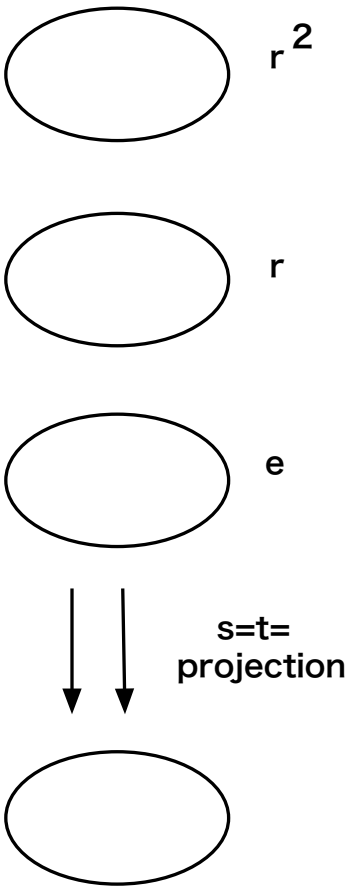
Examples: The Triangular Billiard Groupoid $\mathbb{T}$

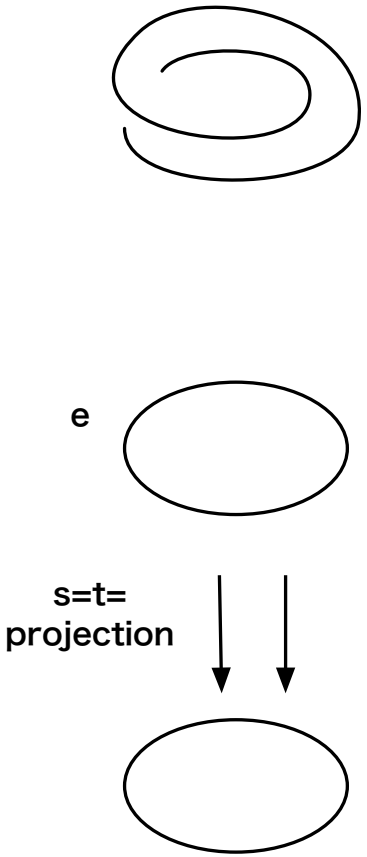




Examples

The $\mathbb{Z}/3$ Circles





Translation Neighbourhoods

Definition

Let $\mathcal{G}$ be an étale and proper topological groupoid.

1. For a point $x \in \mathcal{G}_0$, a neighbourhood $U_x \subseteq \mathcal{G}_0$ is a **translation neighbourhood** if $(s,t)^{-1}(U_x \times U_x) \cong \mathcal{G}_x \times U_x$.
2. For a point $g \in \mathcal{G}_1$, a neighbourhood $U_g \subseteq \mathcal{G}_1$ is a **translation neighbourhood** if $s|U_g$ and $t|U_g$ are both homeomorphisms whose images are translation neighbourhoods.

Lemma (Moerdijk-P, 1997)

When $\mathcal{G}$ is étale and proper, the translation neighbourhoods form a basis for the topology of $\mathcal{G}_0$ and $\mathcal{G}_1$.

Remark

These translation neighbourhoods can be made into an orbi-atlas for $X_{\mathcal{G}}$, but that is the topic of Laura's talk.

Groupoid Homomorphisms

- A **groupoid homomorphism** $f\colon \mathcal{G} \to \mathcal{H}$ is an internal functor in **Top**, i.e., a pair of continuous functions $f_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ and $f_1 : \mathcal{G}_1 \to \mathcal{H}_1$ making the following diagram 'commute':

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "m"] \arrow[d, "f_1 \times f_1"'] & \mathcal{G}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "f_1"'] & \mathcal{G}_0 \arrow[l, "u"'] \arrow[d, "f_0"] \\
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r, "m"] & \mathcal{H}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{H}_0 \arrow[l, "u"']
\end{tikzcd}$$

- There is a space of groupoid homomorphisms

$$\mathbf{GMap}(\mathcal{G}, \mathcal{H})_0 \subset \mathbf{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \mathbf{Map}(\mathcal{G}_1, \mathcal{H}_1)$$

Natural 2-Cells

A **2-cell**

$$\alpha\colon f \Rightarrow f' : \mathcal{G} \rightrightarrows \mathcal{H}$$

is given by an internal natural transformation, i.e., a continuous function

$$\alpha\colon \mathcal{G}_0 \to \mathcal{H}_1$$

such that

- $s \circ \alpha = f_0$ and $t \circ \alpha = f'_0$;
- (naturality) the following square commutes in $\mathcal{H}$ for each $g \in \mathcal{G}_1$,

$$\begin{tikzcd}
f_0(sg) \arrow[r, "f_1(g)"] \arrow[d, "\alpha(sg)"'] & f_0(tg) \arrow[d, "\alpha(tg)"] \\
f'_0(sg) \arrow[r, "f'_1(tg)"'] & f'_0(tg)
\end{tikzcd}$$

The Groupoid $\mathbf{GMap}(\mathcal{G},\mathcal{H})$

- There is a space of natural transformations

$$\mathbf{GMap}(\mathcal{G},\mathcal{H})_1 \subset \mathbf{GMap}(\mathcal{G},\mathcal{H})_0 \times \mathbf{Map}(\mathcal{G}_0,\mathcal{H}_1) \times \mathbf{GMap}(\mathcal{G},\mathcal{H})_0.$$

- The obvious structure maps make $\mathbf{GMap}(\mathcal{G},\mathcal{H})_1$ and $\mathbf{GMap}(\mathcal{G},\mathcal{H})_0$ the space of arrows and the space of objects (respectively) for a topological groupoid $\mathbf{GMap}(\mathcal{G},\mathcal{H})$,

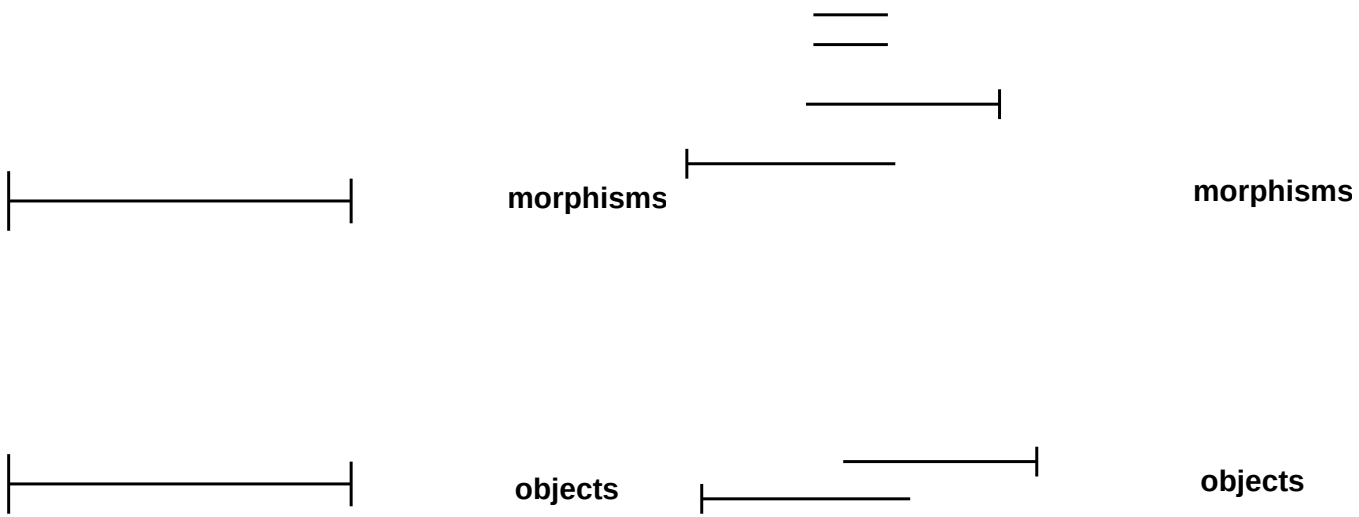
$$\begin{tikzcd} \mathbf{GMap}(\mathcal{G},\mathcal{H})_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbf{GMap}(\mathcal{G},\mathcal{H})_0 \arrow[l, "u"] \end{tikzcd}$$

Examples

- If $f, f' : \mathcal{G} \rightrightarrows \mathcal{H}$ do not induce the same maps on the underlying quotient spaces, there is no 2-cell from f to f'.
- If $\mathcal{G} = G \ltimes X$, a translation groupoid, and $\mathcal{I} = I^{\mathrm{id}}$ then 2-cells $\alpha : f \Rightarrow f' : \mathcal{I} \rightrightarrows \mathcal{G}$ correspond to elements $g \in G$ such that $g \cdot f = f'$.
- It is possible that $f, f' : \mathcal{G} \rightrightarrows \mathcal{H}$ induce the same map on the quotient spaces without being related by a 2-cell.

We do not have enough maps

- The following two representations of the unit interval,



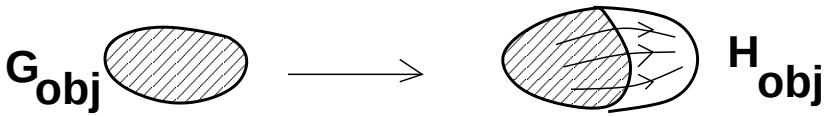
are not isomorphic through groupoid homomorphisms.

Essential Equivalences

- A morphism $f\colon \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is essentially surjective and fully faithful.
- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_0$ in

$$\begin{tikzcd}
\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathcal{H}_0 \\
\mathcal{G}_0 \arrow[r, "f_0"'] & \mathcal{H}_0 &
\end{tikzcd}$$

is an **open surjection**.



f may not be onto the objects of $\mathcal{H}$, but every object in $\mathcal{H}_0$ is isomorphic to an object in the image of $\mathcal{G}_0$.

Essential Equivalences

The morphism $f\colon \mathcal{G} \to \mathcal{H}$ is **fully faithful** when

$$\begin{tikzcd} G_1 \arrow[r, "\phi"] \arrow[d, "(s,t)"'] & H_1 \arrow[d, "(s,t)"] \\ G_0 \times G_0 \arrow[r, "\phi\times\phi"] & H_0 \times H_0 \end{tikzcd}$$

is a pullback,

The local isotropy structure is preserved.

Morita Equivalence

- Two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$ are called **Morita equivalent** if there exists a third orbigroupoid $\mathcal{K}$ with essential equivalences

$$\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}.$$

- This is an equivalence relation on groupoids, because essential equivalences of topological groupoids are stable under weak pullbacks (iso-comma-squares).

Generalized Maps and 2-Cells Between Orbigroupoids

- **Generalized maps** or **orbimaps** are spans

$$\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$$

where υ is an essential equivalence;

- A **2-cell** between two generalized maps is an equivalence class of diagrams of the form

$$\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\upsilon"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[rr, phantom, "\alpha_1 \Downarrow"] & \mathcal{L} \arrow[u, "\nu_1"'] \arrow[d, "\nu_2"] \arrow[rr, phantom, "\alpha_2 \Downarrow"] & \mathcal{H} \\
 & \mathcal{K}' \arrow[ul, "\upsilon'"] \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

where $\upsilon\nu_1$ is an essential equivalence.

Small Hom-Groupoids?

- We want to show that the hom-groupoids of generalized maps and 2-cells between them carry a natural topology that makes them orbispace groupoids.
- Our first obstacle is the fact that the hom-groupoids as just described are not small.
- So we will introduce small subgroupoids that are Morita equivalent to the larger ones.
- This requires the notion of an essential covering map.

Essential Coverings

- A collection $\mathcal{U}$ of open subsets of $\mathcal{G}_0$ is an **essential covering** of $\mathcal{G}_0$ if the map $(j_{\mathcal{U}})_0 \colon \coprod_{U \in \mathcal{U}} U \to \mathcal{G}_0$ is essentially surjective: $t\pi_2$ is an open surjection in

$$\begin{tikzcd}
(\coprod_{U \in \mathcal{U}} U) \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathcal{G}_0 \\
\coprod_{U \in \mathcal{U}} U \arrow[r, "(j_{\mathcal{U}})_0"'] & \mathcal{G}_0 &
\end{tikzcd} .$$

- Note that an essential covering does not necessarily cover all of $\mathcal{G}_0$, but it meets every orbit.

Essential Covering Maps

Any essential covering $\mathcal{U}$ gives rise to a groupoid $\mathcal{G}^*(\mathcal{U})$ with a groupoid homomorphism $j_{\mathcal{U}} \colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$, defined by:

- $\mathcal{G}^*(\mathcal{U})_0 = \coprod_{U \in \mathcal{U}} U$;
- $(j_{\mathcal{U}})_0 \colon \mathcal{G}^*(\mathcal{U})_0 \to \mathcal{G}_0$ is defined by inclusions on the connected components;
- $\mathcal{G}^*(\mathcal{U})_1$ is defined as the pullback,

$$\begin{tikzcd}
\mathcal{G}(\mathcal{U})_1 \arrow[r, "(j_{\mathcal{U}})_1"] \arrow[d, "(s,t)"'] & \mathcal{G}_1 \arrow[d, "(s,t)"] \\
\coprod_{U \in \mathcal{U}} U \times \coprod_{U \in \mathcal{U}} U \arrow[r, "(j_{\mathcal{U}})_0 \times (j_{\mathcal{U}})_0"'] & \mathcal{G}_0 \times \mathcal{G}_0.
\end{tikzcd}$$

- This makes the map $j_{\mathcal{U}} \colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ an essentially equivalence.

Essential Covering Maps

Definition

An essential equivalence $w\colon \mathcal{H} \to \mathcal{G}$ which is isomorphic to one of the form $j_{\mathcal{U}}\colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ as just described is called an **essential covering map**,

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "w"'] \arrow[rr, phantom, shift right=4ex, "\overset{\alpha}{\cong}"] & & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "j_{\mathcal{U}}"] \\
& \mathcal{G} &
\end{tikzcd}.$$

Properties of Essential Covering Maps

- For any groupoid $\mathcal{G}$, there is a **set** of essential covering maps with codomain $\mathcal{G}$.
- For each essential equivalence $\mathcal{H} \xrightarrow{w} \mathcal{G}$ of orbigroupoids there is an essential covering $\mathcal{U}$ of $\mathcal{G}$ such that $j_{\mathcal{U}}$ factors through w,

$$\begin{tikzcd}
\mathcal{G}^*(\mathcal{U}) \arrow[d, "\varepsilon"'] \arrow[dr, "j_{\mathcal{U}}"] & \\
\mathcal{H} \arrow[r, "w"'] \arrow[ur, phantom, "\alpha"] & \mathcal{G}
\end{tikzcd}$$

Essential Coverings and Generalized Maps, I

Lemma

Given two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$, each orbimap

$$\mathcal{G} \xleftarrow[w]{} \mathcal{K} \xrightarrow[\varphi]{} \mathcal{H}$$

is isomorphic to one of the form,

$$\mathcal{G} \xleftarrow[j_{\mathcal{U}}]{} \mathcal{G}^*(\mathcal{U}) \xrightarrow[\psi]{} \mathcal{H}$$

$$\begin{tikzcd}
 & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\psi=\varphi\varepsilon"] \arrow[d, equal] & \\
\mathcal{G} \arrow[r, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{U}) \arrow[r, phantom, "\cong"] \arrow[d, "\varepsilon"'] & \mathcal{H} \\
 & \mathcal{K} \arrow[ul, "w"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Essential Coverings and Generalized Maps, II

Any 2-cell from

$$\mathcal{G} \xleftarrow[j_{\mathcal{U}}]{} \mathcal{G}^*(\mathcal{U}) \xrightarrow[\varphi]{} \mathcal{H}$$

to

$$\mathcal{G} \xleftarrow[j_{\mathcal{V}}]{} \mathcal{G}^*(\mathcal{V}) \xrightarrow[\psi]{} \mathcal{H}$$

can be represented by a diagram of the form

$$\begin{tikzcd}
 & \mathcal{G}^*(\mathcal{U}) \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{W}) \arrow[u, "j_{\mathcal{U}}^{\mathcal{W}}"'] \arrow[d, "j_{\mathcal{V}}^{\mathcal{W}}"] \arrow[r, phantom, "\beta"] & \mathcal{H} \\
 & \mathcal{G}^*(\mathcal{V}) \arrow[ul, "j_{\mathcal{V}}"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

The essential covering $\mathcal{W}$ can be viewed as an essential refinement of $\mathcal{U}$ and $\mathcal{V}$.

The orbi mapping groupoid $\mathbf{OMap}(\mathcal{G},\mathcal{H})$

- Let $\mathbf{OMap}(\mathcal{G},\mathcal{H})_0$ be the space generalized maps from $\mathcal{G}$ to $\mathcal{H}$. This is topologized as the following coproduct of spaces,

$$\mathbf{OMap}(\mathcal{G},\mathcal{H}) = \coprod_{(\mathcal{U},j_{\mathcal{U}})} \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{H})_0$$

- The space $\mathbf{OMap}(\mathcal{G},\mathcal{H})_1$ is the space of equivalence classes of 2-cell diagrams, so it can be viewed as a quotient of a subspace of a product of mapping spaces.

Pseudo Colimits

Theorem (P-Scull)

For any two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$, the groupoid **OMap**$(\mathcal{G},\mathcal{H})$ *of generalized maps and 2-cells between them, is Morita equivalent to the pseudo colimit*

$$\varinjlim_{(\mathcal{U},j_{\mathcal{U}})} \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}),\mathcal{H}).$$

This colimit is taken over the diagram of essential coverings of $\mathcal{G}$ with essential refinements between them,

$$\begin{tikzcd}
\mathcal{G}^*(\mathcal{U}) \arrow[rr, "j_{\mathcal{V}}^{\mathcal{U}}"] \arrow[dr, "j_{\mathcal{U}}"'] & \arrow[d, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{V}) \arrow[dl, "j_{\mathcal{V}}"] \\
& \mathcal{G} &
\end{tikzcd}$$

Generalized 2-Cells

Given a common essential refinement,

$$\begin{tikzcd} \mathcal{G}^*(\mathcal{W}) \arrow[r, "j^{\mathcal{U}}_{\mathcal{W}}"] \arrow[d, "j^{\mathcal{V}}_{\mathcal{W}}"'] \arrow[dr, phantom, "\alpha\cong"] & \mathcal{G}^*(\mathcal{U}) \arrow[d, "j_{\mathcal{U}}"] \\ \mathcal{G}^*(\mathcal{V}) \arrow[r, "j_{\mathcal{V}}"'] & \mathcal{G} \end{tikzcd}$$

every generalized 2-cell between any pair of orbimaps,

$$\mathcal{G} \xleftarrow{j_{\mathcal{U}}} \mathcal{G}^*(\mathcal{U}) \xrightarrow{\varphi} \mathcal{H} \quad \text{and} \quad \mathcal{G} \xleftarrow{j_{\mathcal{V}}} \mathcal{G}^*(\mathcal{V}) \xrightarrow{\psi} \mathcal{H}$$

can be represented by a diagram of the form

$$\begin{tikzcd} & \mathcal{G}^*(\mathcal{U}) \arrow[dl] \arrow[dr] & \\ \mathcal{G} \arrow[r, phantom, "\alpha"] & \mathcal{G}^*(\mathcal{W}) \arrow[u] \arrow[d] \arrow[r, phantom, "\beta"] & \mathcal{H} \\ & \mathcal{G}^*(\mathcal{V}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The Space of Arrows for $\mathbf{OMap}(\mathcal{G},\mathcal{H})$

Let $\mathcal{G}$ and $\mathcal{H}$ be orbigroupoids. Then the space of arrows of the mapping groupoid for orbimaps can be described as follows:

- For every pair of essential coverings $\mathcal{U}_i$, $\mathcal{U}_j$ of $\mathcal{G}$, choose an essential common refinement,

$$\begin{tikzcd}
\mathcal{G}^*(\mathcal{W}_{ij}) \arrow[r, "j^{\mathcal{U}_i}_{\mathcal{W}_{ij}}"] \arrow[d, "j^{\mathcal{U}_j}_{\mathcal{W}_{ij}}"'] \arrow[dr, phantom, "\alpha_{ij}\cong"] & \mathcal{G}^*(\mathcal{U}_i) \arrow[d, "j_{\mathcal{U}_i}"] \\
\mathcal{G}^*(\mathcal{U}_j) \arrow[r, "j_{\mathcal{U}_j}"'] & \mathcal{G}
\end{tikzcd}$$

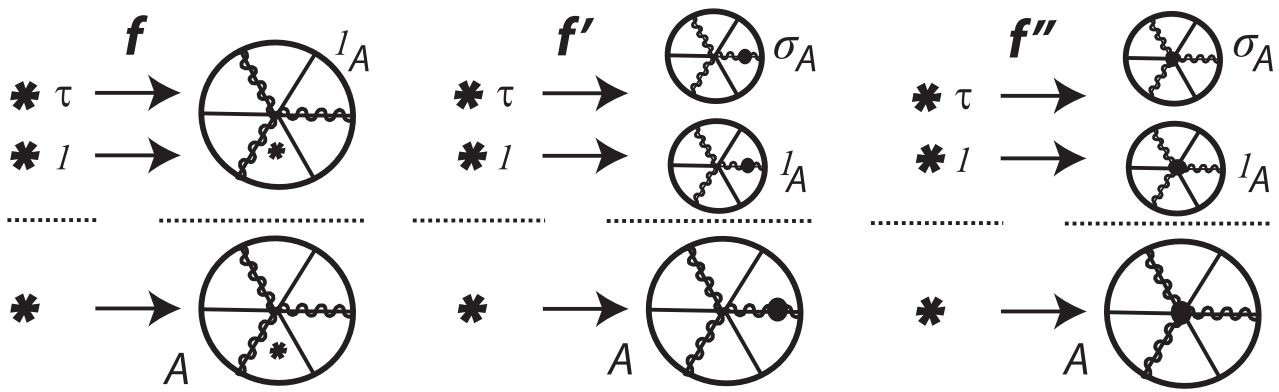
- $\mathbf{OMap}(\mathcal{G},\mathcal{H})_1$ can be viewed as

$$\coprod_{i,j} (j^{\mathcal{U}_i}_{\mathcal{W}_{ij}}, j^{\mathcal{U}_j}_{\mathcal{W}_{ij}})^* \mathbf{GMap}(\mathcal{G}^*(\mathcal{W}_{ij}),\mathcal{H})_1 \subset$$

$$\coprod_{i,j} \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}_i),\mathcal{H})_0 \times \mathbf{Map}(\mathcal{G}^*(\mathcal{W}_{ij})_0,\mathcal{H}_1) \times \mathbf{GMap}(\mathcal{G}^*(\mathcal{U}_j),\mathcal{H})_0$$

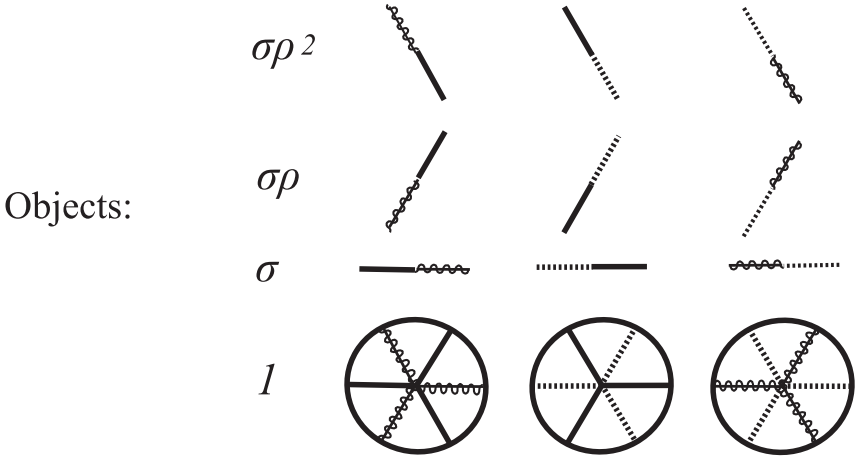
Example: The $\mathbb{Z}/2$-points of the triangular billiard

- Let $*_{\mathbb{Z}/2} = \mathbb{Z}/2 \ltimes \{*\}$. We will calculate $\mathbf{OMap}(*_{\mathbb{Z}/2}, \mathbb{T})$.
- No essential coverings are needed in this case: $\mathbf{OMap}(*_{\mathbb{Z}/2}, \mathbb{T}) = \mathbf{GMap}(*_{\mathbb{Z}/2}, \mathbb{T})$.
- Here are the options for maps from $*_{\mathbb{Z}/2}$ to the triangular billiard groupoid $\mathbb{T}$,

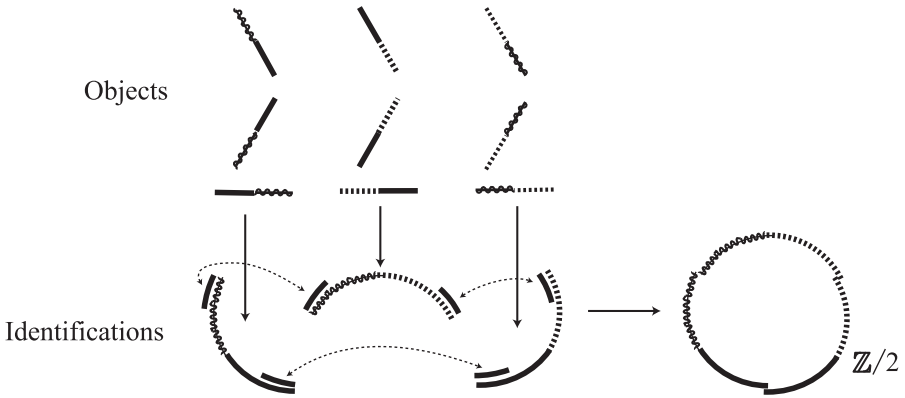


Example: The $\mathbb{Z}/2$-points of the triangular billiard

- The space of objects becomes

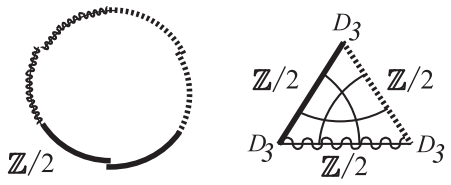


- Some of the natural transformations



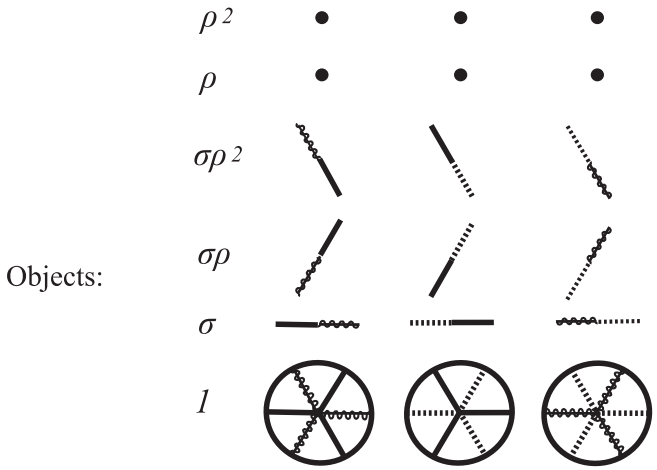
Example: The $\mathbb{Z}/2$-points of the triangular billiard

- The final result is that we obtain a copy of the original orbigroupoid $\mathbb{T}$ together with a copy of the (trivial) $\mathbb{Z}/2$-circle, $S^1_{\mathbb{Z}/2}$,

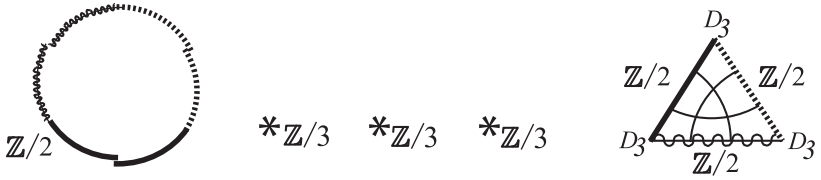


Example: The $\mathbb{Z}/6$-points of the triangular billiard

- The space of objects,



- The resulting orbispace,



The Inertia Groupoid

Proposition

For any orbispace with a compact underlying space and groupoid representation $\mathcal{G}$, there is an integer n such that $\Lambda(\mathcal{G}) = \mathbf{OMap}(*_{\mathbb{Z}/n}, \mathcal{G}) = \mathbf{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$.

Results So Far

Theorem (P-Scull)

- *For an orbit compact orbigroupoid $\mathcal{G}$ and any orbigroupoid $\mathcal{H}$ the groupoid* $\mathbf{OMap}(\mathcal{G},\mathcal{H})$ *is again étale and proper;*
- *This construction of* $\mathbf{OMap}(\mathcal{G},\mathcal{H})$ *is functorial in $\mathcal{G}$ and $\mathcal{H}$ for generalized maps;*
- *A Morita equivalence $\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{G}'$ induces an isomorphism*

$$\mathbf{OMap}(\mathcal{G},\mathcal{H}) \cong \mathbf{OMap}(\mathcal{G}',\mathcal{H});$$

- *A Morita equivalence $\mathcal{H} \leftarrow \mathcal{K} \rightarrow \mathcal{H}'$ induces an isomorphism*

$$\mathbf{OMap}(\mathcal{G},\mathcal{H}) \cong \mathbf{OMap}(\mathcal{G},\mathcal{H}').$$