An Atlas Definition for Noneffective Orbifolds

Laura Scull
joint with Dorette Pronk and Matteo Tommasini

Outline

Effective Orbifolds: Atlases and Groupoids
- Effective Orbifold Atlases
- Effective Orbifolds via Groupoids

Noneffective Orbifolds
- Noneffective Orbifold Atlases
- New Orbifold Atlases

What is an orbifold?

- a space that is locally modelled by quotients of $\mathbb{R}^n$ by actions of finite groups
- a generalization of a manifold
- allows controlled singularities

Orbifold Charts

A (classical) **orbifold chart** is given by a connected open subset $U \subseteq X$ and a uniformizing system $(\tilde{U}, G_U, \varphi_U)$, where

1. $\tilde{U}$ is an open subset of $\mathbb{R}^n$;
2. G_U is a finite group acting (smoothly) on $\tilde{U}$;
3. $\varphi_U \colon \tilde{U} \to U$ gives a homeomorphism $\tilde{U}/G_U \xrightarrow{\sim} U$.

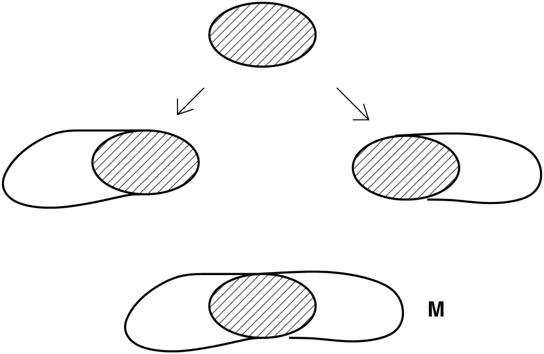
Satake Embeddings

For two charts $U \subseteq V$ a Satake embedding $\lambda\colon (\tilde{U}, G_U, \varphi_U) \to (\tilde{V}, G_V, \varphi_V)$ is a smooth embedding $\lambda\colon \tilde{U} \to \tilde{V}$ such that

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\lambda} & \tilde{V} \\ \downarrow{\scriptstyle \varphi_U} & & \downarrow{\scriptstyle \varphi_V} \\ U & \subseteq & V. \end{array}$$

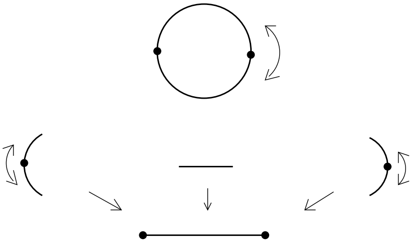
Orbifold Atlas

An atlas is a covering of M by orbifold charts such that if $x \in U \cap V \subseteq M$, there is a chart with $x \in W \subseteq U \cap V$ and atlas chart embeddings $U \hookleftarrow W \hookrightarrow V$



Examples

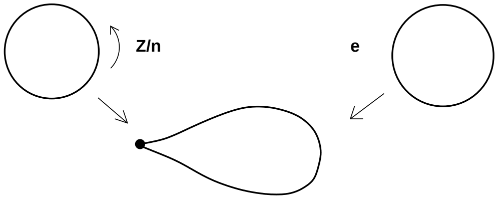
$M = S^1$ with $G = \mathbb{Z}/2$ action



The orbifold consists of the orbit space M/G together with the data about the isotropy groups.

Examples

[Thurston] The teardrop orbifold:



Atlas Chart Embeddings

Observations: [Satake]

Let $\mathcal{U} = \{\tilde{U}, G_U, \varphi_U\}$ and $\mathcal{V} = \{\tilde{V}, G_V, \varphi_V\}$ be charts.

- For any two chart embeddings $\lambda, \mu : \widetilde{U} \hookrightarrow \widetilde{V}$, there exists a unique $g' \in G_V$ such that $\mu = g'\lambda$.
- Given any $\lambda : \widetilde{U} \hookrightarrow \widetilde{V}$ and $g \in G_U$, there is a unique $g' \in G_V$ such that $\lambda g = g'\lambda$.

Describing Orbifolds with Categories

We can also encode orbifold information via groupoids:

a space of objects G_0

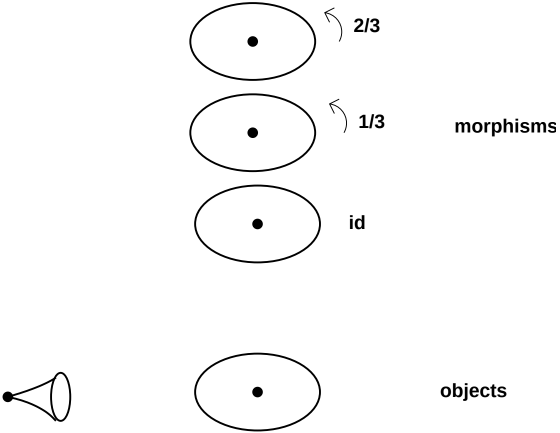
a space of identifications G_1

all structure maps smooth

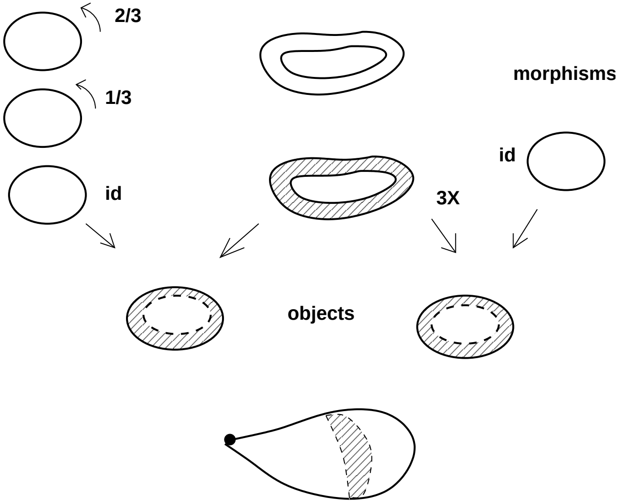
$$M = G_0/G_1 = G_0/(s(g_1) \sim t(g_1)), g_1 \in G_1$$

An orbigroupoid is Lie, étale and proper.

Orbifold Groupoids



Teardrop Orbifold



Morita Equivalence

- Two orbigroupoids represent the same orbifold if they are Morita Equivalent.
- Generated by functors $\phi : \mathcal{G} \to \mathcal{H}$ which are essentially surjective and fully faithful in a suitably topologized sense.
- (Pronk, 1995)

Atlas for effective orbifold up to common refinement	$\longleftrightarrow$	Effective orbigroupoid up to Morita equiv

Atlas Groupoids

Given an orbifold atlas $\mathfrak{A}$, we define the topological category

- space of objects $C(\mathfrak{A})_0 = \coprod_{\mathcal{U}\in\mathfrak{A}} \tilde{U}$;
- space of arrows

$$C(\mathcal{U})_1 = \coprod_{\lambda\colon \tilde{U}\to\tilde{V}} \tilde{U} = \bigcup_{\lambda\colon \tilde{U}\to\tilde{V}} \{\lambda\} \times \tilde{U}$$

To get a groupoid $\mathcal{G}(\mathcal{U})$, we invert the morphisms via category of fractions.

Noneffective orbifolds

An orbifold is *noneffective* (ineffective, non-reduced) if the group actions are not effective.

These arise naturally:

- moduli stacks
- string cohomology
- orbispaces as a category

To get noneffective orbigroupoids, drop the effective condition.

Atlases for Noneffective Orbifolds

Let U be a non-empty connected topological space; an *orbifold chart* for U is a quadruple $(\widetilde{U}, G, \rho, \varphi)$ where:

- $\widetilde{U}$ is a connected open subset of $\mathbb{R}^n$;
- G is a finite group;
- $\rho : G \to \mathrm{Aut}(\widetilde{U})$ is a representation of G as a group of smooth automorphisms of $\widetilde{U}$; we set $G^{\mathrm{red}} := \rho(G) \subseteq \mathrm{Aut}(\widetilde{U})$ and $\ker(G) := \ker(\rho) \subseteq G$;
- $\varphi : \widetilde{U} \to U$ is a homeomorphism between U and $\widetilde{U}/G^{\mathrm{red}}$.

Atlas Chart Embeddings

Earlier Observations: Let $\mathcal{U} = \{\tilde{U}, G_U, \varphi_U\}$ and $\mathcal{V} = \{\tilde{V}, G_V, \varphi_V\}$ be charts.

- For any two chart embeddings $\lambda, \mu : \widetilde{U} \hookrightarrow \widetilde{V}$, there exists a unique $g' \in G_V$ such that $\mu = g'\lambda$.
- Given any $\lambda : \widetilde{U} \hookrightarrow \widetilde{V}$ and $g \in G_U$, there is a unique $g' \in G_V$ such that $\lambda g = g'\lambda$.
- The correspondence $g \to g'$ gives a homomorphism $\ell : G_U \to G_V$ such that $\lambda(gu) = \ell(g)\lambda(u)$ for $u \in \widetilde{U}$
- The homomorphism $\ell : G_U \to G_V$ is injective

Chart Embeddings for Noneffective Orbifolds

Given charts $\mathcal{U} = \{\tilde{U}, G_U, \rho_U, \varphi_U\}$ and $\mathcal{V} = \{\tilde{V}, G_V, \rho_V, \varphi_V\}$, an (atlas) chart embedding $\mathcal{U} \hookrightarrow \mathcal{V}$ consists of a pair

$$(\lambda\colon \tilde{U} \hookrightarrow \tilde{V}, \ell\colon G_U \hookrightarrow G_V)$$

such that

$$\begin{tikzcd} \tilde{U} \arrow[r, "\lambda"] \arrow[d, "\varphi_U"'] & \tilde{V} \arrow[d, "\varphi_V"] \\ U \arrow[r, phantom, "\subseteq"] & V \end{tikzcd}$$

$\lambda(g \cdot u) = \ell(g) \cdot \lambda(u)$ for $g \in G_U$ and $u \in \tilde{U}$, and

$\ell|_{\mathrm{Ker}(\rho_U)} : \mathrm{Ker}(\rho_U) \xrightarrow{\sim} \mathrm{Ker}(\rho_V)$.

The Problem

Consider the orbifold described by one chart D, the open unit disk in $\mathbb{R}^2$, with $G_D = \mathbb{Z}/3$, acting trivially.

Chart embeddings from D to itself:

$G_D = \mathrm{Ker}(\rho_D)$, so ℓ needs to be an automorphism $\mathbb{Z}/3 \to \mathbb{Z}/3$.

There are two possible automorphisms.

If we make the groupoid out of this very small atlas, we will get a disk with noneffective $\mathbb{Z}/2$ action

Goals of Our Project

We want to create a new definition of atlas for orbifolds such that:

- For effective orbifolds, it is equivalent to the classical definition.
- For noneffective orbifolds, it will correspond to the orbigroupoid definition.

Atlas Chart Embeddings

Earlier Observations: Let $\mathcal{U} = \{\tilde{U}, G_U, \varphi_U\}$ and $\mathcal{V} = \{\tilde{V}, G_V, \varphi_V\}$ be charts.

- For any two chart embeddings $\lambda, \mu : \widetilde{U} \hookrightarrow \widetilde{V}$, there exists a unique $g' \in G_V$ such that $\mu = g'\lambda$.
- Given any $\lambda : \widetilde{U} \hookrightarrow \widetilde{V}$ and $g \in G_U$, there is a unique $g' \in G_V$ such that $\lambda g = g'\lambda$.
- The group G_U has a right action and the group G_V has a left action by composition on the set of Satake embeddings $\mathrm{Emb}(\tilde{U}, \tilde{V})$.
- Both actions are free; G_V is transitive.

Atlas Modules

Given two groups G and H, a set M with a left H and right G action is called an H-G **atlas module** when:

1. The actions are compatible: $(h \cdot m) \cdot g = h \cdot (m \cdot g)$;
2. Both G and H act freely.
3. H acts transitively.

Remark

- This makes M into an H-torsor, so for a finite group H, $\#(H) = \#(M)$.
- For effective atlases the set $\mathrm{Emb}(\tilde{U}, \tilde{V})$ is a G_V-G_U atlas bimodule.

2 Level Modules: Abstract and Concrete

- The Satake embeddings $\mathrm{Emb}(\tilde{U}, \tilde{V}) = \mathrm{Con}(U, V)$ form a G_V^{red}-G_U^{red} atlas bimodule; call this the concrete bimodule
- we will specify that an orbifold atlas contains an atlas G_V-G_U bimodule
$$\mathrm{Abst}(U, V)$$
of (abstract) embeddings
- We require compatibility between levels
- We describe the compatibility via the 2-category GroupMod of bimodules and pseudo functors

The New Atlas Definition

An *orbifold atlas* of dimension n for X is the datum of:

(1) $\mathcal{O}(\mathcal{U})$ a cover of X by open sets, interepreted as a category with inclusion maps

(2) $\mathcal{U} = \{(\widetilde{U}_i, G_i, \rho_i, \pi_i)\}_{i\in I}$ orbifold charts of dimension n, such that $\{(\widetilde{U}_i, G_i^{\text{red}}, \pi_i)\}_{i\in I}$ form an effective atlas with pseudo functor $(\text{Con}, \gamma)\colon \mathcal{O}(\mathcal{U}) \to \text{GroupMod}$;

(3) a pseudofunctor $\text{Abst} : \mathcal{O}(\mathcal{U}) \longrightarrow \text{GroupMod}$ such that the set of embeddings $\text{Abst}(\mu_{ji})$ is an atlas bimodule $G_i \nrightarrow G_j$

The New Atlas Definition Cont'd

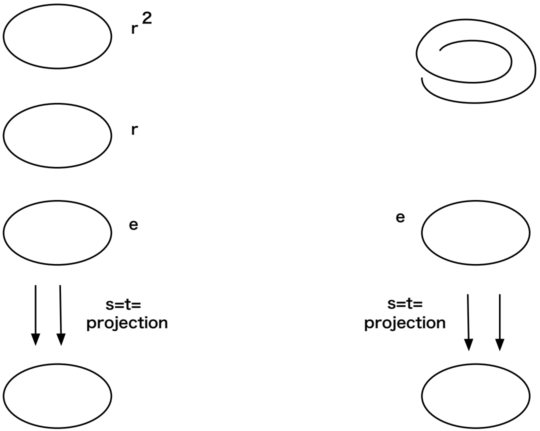
(4) Compatibility: a surjective oplax transformation $\boldsymbol{\rho} = (\{\boldsymbol{\rho}_i\}_{i\in I}, \{\rho_{ji}\}_{i,j\in I})\colon \mathrm{Abst} \Rightarrow \mathrm{Con}$ so that for each arrow $U_i \xrightarrow{\mu_{ji}} U_j$ in $\mathcal{O}(\mathcal{U})$, there is a map of bimodules,

$$\boldsymbol{\rho}_{ji} : \boldsymbol{\rho}_j \otimes_{G_j} \mathrm{Abst}(\mu_{ji}) \Longrightarrow \mathrm{Con}(\mu_{ji}) \otimes_{G_i^{\mathrm{red}}} \boldsymbol{\rho}_i,$$

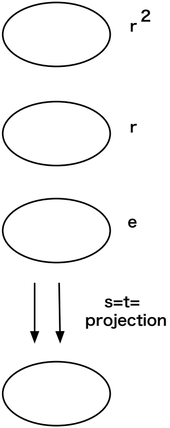
$$\begin{tikzcd} G_i \arrow[r, "\mathrm{Abst}(\mu_{ji})"] \arrow[d, "\boldsymbol{\rho}_i"'] \arrow[dr, phantom, "\Downarrow\rho_{ji}"] & G_j \arrow[d, "\boldsymbol{\rho}_j"] \\ G_i^{\mathrm{red}} \arrow[r, "\mathrm{Con}(\mu_{ji})"] & G_j^{\mathrm{red}} \end{tikzcd}$$

(5) Transitivity on the kernel: if $\rho_{ji}(e_j^{\mathrm{red}} \otimes \lambda) = \rho_{ji}(e_j^{\mathrm{red}} \otimes \lambda')$ for $\lambda, \lambda' \in \mathrm{Abst}(\mu_{ji})$, there is an element $g \in G_i$ such that $\lambda g = \lambda'$

Examples: Purely Noneffective

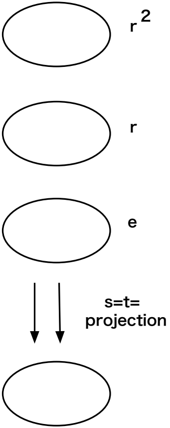


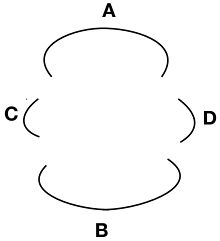
Examples: Untwisted



$$
\begin{tikzcd}
Abst(U) = G_U = \mathbb{Z}/3 \arrow[d] \\
Con(U) = G_U^{red} = \{*\}
\end{tikzcd}
$$

Example: Untwisted





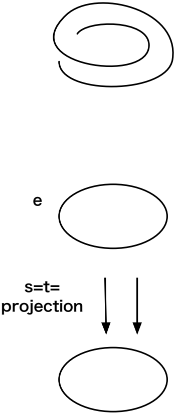
Inclusion C to A:

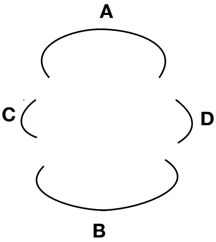
$$\begin{tikzcd} \mathbb{Z}/3 \arrow[r, "M"] \arrow[d] & \mathbb{Z}/3 \arrow[d] \\ \{1\} \arrow[r] & \{1\} \end{tikzcd}$$

$$M = \mathbb{Z}/3$$

$$\begin{array}{ll} \rho 1 = r & 1\rho = r \\ \rho r = r^2 & r\rho = r^2 \\ \rho r^2 = 1 & r^2\rho = 1 \end{array}$$

Example: Twisted





Inclusion C to A:

$$\begin{tikzcd} \mathbb{Z}/3 \arrow[r, "M"] \arrow[d] & \mathbb{Z}/3 \arrow[d] \\ \{1\} \arrow[r] & \{1\} \end{tikzcd}$$

$M = \mathbb{Z}$

$$\begin{array}{ll} \rho 1 = r & 1\rho = r^2 \\ \rho r = r^2 & r\rho = 1 \\ \rho r^2 = 1 & r^2\rho = r \end{array}$$

Results

Theorem (Pronk-S-Tommasini)

The atlas groupoid constructed from an orbifold atlas (in the sense of the previous definition) is an orbigroupoid (ie étale, proper).

We use a topological version of the Gabriel-Zisman construction of the category of fractions to form the groupoid from the atlas category.

Theorem (Pronk-S-Tommasini)

Given an orbigroupoid, we can create an orbifold atlas (in the sense of the previous definition) using translation neighbourhoods.

Theorem (Pronk-S-Tommasini)

Two atlas structures are the same up to common refinement if and only if the corresponding orbigroupoids are Morita equivalent.

Reference

Atlas Structures for Noneffective Orbifolds D. Pronk, L. Scull, M. Tommasini, *almost done!*