Duality of gerbes on orbifolds

Xiang Tang

Memphis, TN, October 18th, 2015

In this talk, we explain a duality conjecture about gerbes on orbifolds using noncommutative geometry. The "Mackey machine" of induced representations provides a powerful approach to study this conjecture.

This is joint work with Ilya Shapiro and Hsian-hua Tseng.

Plan:

1. Group extensions

2. Duality of gerbes

Part I: Group Extension

We review the Mackey machine on finite groups. Such an idea of induced representations goes back to Frobenius and Schur.

G Extension of a finite group Q

Consider an exact sequence of finite groups

$$1 \to G \xrightarrow{i} H \xrightarrow{j} Q \to 1. \tag{1}$$

Choose a section $s : Q \to H$. As G is normal in H, the group H acts on G by conjugation. The section s defines a Q "action" α on G by

$$\alpha(q)(g) = s(q)gs(q)^{-1}.$$

Generally this is not a real group action because $\alpha(q_1) \circ \alpha(q_2) \neq \alpha(q_1 q_2)$.

Define $c : Q \times Q \to G$ by $c(q_1, q_2) = s(q_1)s(q_2)s(q_1 q_2)^{-1}$. The the failure of α being a group action can be computed by

$$\alpha(q_1) \circ \alpha(q_2) = Ad_{c(q_1,q_2)} \circ \alpha(q_1 q_2).$$

Mackey machine I

Mackey machine provides an algorithm to determine H representations in terms of representations of G and Q.

Let V be a representation of H. The group G as a normal subgroup of H also acts on V. As G is finite, the space V as a G representation is naturally decomposed into a direct sum of irreducible G representations, i.e.

$$V = \bigoplus_{\rho \in \widehat{G}} V_\rho, \tag{2}$$

where $\widehat{G}$ is the set of isomorphism classes of irreducible representations of G and V_ρ a subspace of V is a direct sum of G sub-representations of V which are isomorphic to ρ.

Mackey machine II

As G is a normal subgroup of H, the group H acts on G by conjugation. Accordingly, H acts on $\widehat{G}$ the set of isomorphism classes of G irreducible representations. This induces a $Q = H/G$ action on $\widehat{G}$ as G acts on $\widehat{G}$ trivially.

Let χ_ρ be the orthogonal projection from V to V_ρ. Then we have

$$T_h \chi_\rho T_h^{-1} = \chi_{h(\rho)}.$$

If V is an H irreducible representation, then the elements in $\widehat{G}$ appearing in V forms an orbit of the Q action on $\widehat{G}$.

Let θ be an orbit of the Q action on $\widehat{G}$. Choose $\rho \in \theta$, and let Q_θ be the stabilizer subgroup of ρ. Irreducible representations of H associated to θ are one to one correspondent to τ_θ-twisted irreducible representations of the group Q_θ.

$U(1)$ **valued cocycle**

Choose $\rho \in \theta$. The group Q_θ keeps ρ invariant, i.e. $q(\rho) = \rho$, for $q \in Q_\theta$.

Write V_ρ as $E_\rho \otimes F_\rho$, where E_ρ is a G irreducible representation of the isomorphism class ρ. Since $T_q \chi_\rho T_q^{-1} = \chi_{q(\rho)} = \chi_\rho$ for $q \in Q_\theta$, the group Q_θ preserves V_ρ, i.e. $T_q(V_\rho) \subset V_\rho$ for $q \in Q_\theta$.

As the G action on E_ρ is irreducible, we have that T_q acts on $V_\rho = E_\rho \otimes F_\rho$ diagonally, i.e. $T_q = E_q \otimes F_q$ for $q \in Q_\theta$. It is easy to check that $c(q_1, q_2) E_{s(q_1 q_2)} E_{q_2}^{-1} E_{q_1}^{-1}$ commutes with the G action on E_ρ. Therefore, $\tau_\theta(q_1, q_2) = c(q_1, q_2) E_{s(q_1 q_2)} E_{q_2}^{-1} E_{q_1}^{-1}$ is an element of $U(1)$.

The group Q_θ acts on F_ρ with cocycle τ_θ.

Duality suggested by Mackey machine

The Mackey machine suggests the following Morita equivalence result.

The group algebra $\mathbb{C}H$ is Morita equivalent to the algebra

$$\bigoplus_{\theta} \mathbb{C}_{\tau_\theta} Q_\theta.$$

Namely, the category of H modules is isomorphic to the category of $\bigoplus\limits_{\theta} \mathbb{C}_{\tau_\theta} Q_\theta$ modules.

Geometrically, this suggests that there is some duality between the group H and the action groupoid $\widehat{G} \rtimes Q$ with a $U(1)$-valued groupoid 2-cohomology class $[\tau]$.

A toy example

When Q is the trivial group, then $H = G$. Our Morita equivalence result states that the group algebra $\mathbb{C}G$ is Morita equivalent to $C(\widehat{G})$, the algebra of functions on $\widehat{G}$.

This is a corollary of the classical result that $\mathbb{C}G$ is isomorphic to

$$\bigoplus_{\rho \in \widehat{G}} \operatorname{End} E_\rho. \tag{3}$$

In particular, if G is abelian, then $\mathbb{C}G$ is isomorphic to $C(\widehat{G})$ via the Fourier transform.

Our discussion today is a generalization of the classical "Pontryagin duality" on abelian groups to gerbes over orbifolds.

Part II: Duality of gerbes

We explain the duality conjecture of gerbes on orbifolds, and present a number of mathematical results realizing such a principle of duality.

Global quotient

Let H act on a smooth manifold M such that G acts on M trivially. Accordingly, the group $Q = H/G$ acts on M. We have the following exact sequence of action groupoids,

$$M \times G \to M \rtimes H \to M \rtimes Q. \tag{4}$$

As is explained before, Q acts on $\widehat{G}$, the set of isomorphism classes of G irreducible representations. Define a new groupoid $\mathfrak{Q} = (\widehat{G} \times M) \rtimes Q$. The cocycle $[\tau]$ defines a groupoid cocycle $[\tau]$ on $\mathfrak{Q}$.

The Mackey machine suggests that there is a duality between the groupoid extension (4) and the groupoid $\mathfrak{Q}$ with the cocycle $[\tau]$. This suggests a general duality about gerbes on orbifolds.

Duality conjecture

Consider the G-gerbe $X = [M/H]$ and the orbifold $Y = [\widehat{G} \times M/Q]$ with the $U(1)$ valued class $[\tau]$.

Hellerman, Henriques, Pantev, and Sharpe conjectured that the conformal field theory on the gerbe X is same as the conformal field theory on Y with the $U(1)$-gerbe defined by $[\tau]$.

Our viewpoint toward this conjecture is that it suggests the following principle:

Geometry/topology of the G-gerbe X is equivalent to geometry/topology of Y twisted by $[\tau]$.

Morita equivalence

The following results are outcomes of the Mackey machine.

Theorem 1 *The crossed product algebra* $C^\infty(M) \rtimes H$ *is Morita equivalent to the* τ*-twisted crossed product* $C^\infty(\widehat{G} \times M) \rtimes_\tau Q$.

Furthermore, if M is equipped with a K invariant symplectic form, then we consider a K invariant deformation quantization $\mathcal{A}^{((\hbar))}(M)$ of the algebra of smooth functions on M.

Theorem 2 *The crossed product algebra* $\mathcal{A}^{((\hbar))}(M) \rtimes H$ *is Morita equivalent to the* τ *twisted crossed product algebra* $\mathcal{A}^{((\hbar))}(\widehat{G} \times M) \rtimes_\tau Q$.

Hochschild Cohomology

The above Morita equivalence between algebras imply that their corresponding Hochschild and cyclic cohomology groups are isomorphic.

The Hochschild cohomology of $\mathcal{A}^{((\hbar))}(M) \rtimes H$ isomorphic to the cohomology of the space $\coprod_{<h>} M^h/C(h)$.

The Hochschild cohomology of $\mathcal{A}^{((\hbar))}(\widehat{G} \times M) \rtimes_\tau Q$ is computed by τ twisted de Rham cohomology on $\coprod_{<q>} (\widehat{G} \times M)^q/C(q)$.

Theorem 3 *The cohomology rings of the orbifolds $\coprod_{<h>} M^h/C(h)$ and $\coprod_{<q>}(\hat{G} \times M)^q/C(q)$ are isomorphic.*

$$H^\bullet(\coprod_{<h>} M^h/C(h); \mathbb{C}) \simeq H^\bullet(\coprod_{<q>} (\widehat{G} \times M)^q/C(q), \tau; \mathbb{C})$$

Quantum Cohomology

Using the theorem about the isomorphism between the cohomology rings, we can prove the following theorem about quantum cohomology and Gromov-Witten invariants.

Theorem 4 *When X satisfies one of the following properties,*

- *X is a G-gerbe over BQ,*
- *X is a G-gerbe over a manifold with a trivial band,*
- *X is a toric gerbe over a toric orbifold,*

the Gromov-Witten theories of X and Y are isomorphic.

We are working on the following theorem.

When X is a G-gerbe over an orbifold with a trivial band, the Gromov-Witten theories of X and Y are isomorphic.

Remarks

1. Following from the Morita equivalence, we have an isomorphism of K-groups between the two orbifolds.

2. Consider categories of sheaves on $X = [M/H]$ and $[\widehat{G} \times M/Q]$. The category of sheaves on $[M/H]$ is isomorphic to the category of τ-twisted sheaves on $Y = [\widehat{G} \times M/Q]$.

3. If $\mathfrak{t}$ is an S^1-valued 2-cocycle on G, let $G_{\mathfrak{t}}$ be the central extension of G by S^1 via $\mathfrak{t}$. We can generalize our results to the extension

$$1 \longrightarrow G_{\mathfrak{t}} \longrightarrow H_{\mathfrak{t}} \longrightarrow Q \longrightarrow 1.$$

Thank you!