Haar Systems on Equivalent Groupoids

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Haar Systems

- In order to make a C^*-algebra from a groupoid G, we need a family of Radon measures $\{\, \lambda^u : u \in G^{(0)} \,\}$ such that
- $\operatorname{supp} \lambda^u = G^u = \{\, x \in G : r(x) = u \,\}$ and such that
- for all $f \in C_c(G)$,

$$u \mapsto \int_G f(x) \, d\lambda^u(x)$$

is continuous, and
- for all $f \in C_c(G)$,

$$\int_G f(yx) \, d\lambda^{r(y)}(x) = \int_G f(x) \, d\lambda^{s(y)}(x).$$

(Alternatively, $x \cdot \lambda^{s(x)} = \lambda^{r(x)}$ where
$x \cdot \lambda^{s(x)}(E) = \lambda^{s(x)}(x^{-1}E)$.)
- We call $\{\lambda^u\}$ a Haar system on G.

- If $\mathcal{G}$ is a locally compact group, then it always has a Haar measure.
- Even better, any two Haar measures on $\mathcal{G}$ differ by a positive scalar. So Haar measure on a group is as unique as it can be.
- This raises the obvious questions: Does every locally compact groupoid have a Haar system? and
- If λ and λ' are Haar systems on G, how are they related?.

Uniqueness

Example (Not much Uniqueness)

- Let $G = X \times X$ be the "pair groupoid": $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$.
- Then $\lambda^x = \delta_x \times \mu$ gives us a Haar system for any measure μ on X with full support.

Theorem (MRW, '87)

If λ and λ' are Haar systems on a second countable locally compact groupoid G, then $C^*(G, \lambda)$ and $C^*(G, \lambda')$ are Morita equivalent.

Question

Are $C^*(G, \lambda)$ and $C^*(G, \lambda')$ always isomorphic?

Remark

If G has a Haar system, then the range (and hence the source) map is open. Since there do exist locally compact groupoids where the range map fails to be open, we will have to add the assumption if we want to produce a Haar system.

Question

Does every second countable locally compact groupoid with open range map have a Haar system?

Example

- G is called étale if in addition to having open range and source maps, $G^{(0)}$ is open in G. Then each G^u is discrete and the appropriate collection of counting measures forms a Haar system.

- Lie groupoids always have Haar systems.

- In order for a groupoid to act on a space Z, we need a moment map $r : Z \to G^{(0)}$.

- Then a G-action consists of a continuous map $(g, z) \mapsto g \cdot z$ from $G * Z = \{ (g, z) : s(g) = r(z) \}$ to G such that $r(z) \cdot z = z$ and $(gh) \cdot z = g \cdot (h \cdot z)$ if $(g, h) \in G^{(2)}$.

- It goes without saying that we want the moment map to be continuous. In this talk, it will also usually be required to be open.

- The action is called free if $g \cdot z = z$ implies $g = r(z)$.

- The action is called proper if the map $(g, z) \mapsto (g \cdot z, z)$ is proper as a map from $G * Z$ to $Z \times Z$.

Equivalence

Definition (Groupoid Equivalence)

Recall that two groupoids G and H are equivalent if there is a space Z which is a free and proper left G-space and a free and proper right H-space such that the actions commute, the moment map $r : Z \to G^{(0)}$ factors through a homeomorphism of Z/H onto $G^{(0)}$, and the moment map $s : Z \to H^{(0)}$ factors through a homeomorphism of $G \backslash Z$ onto $H^{(0)}$.

Example

We say that G is transitive if given $u, v \in G^{(0)}$ there is a $x \in G$ such that $s(x) = u$ and $r(x) = v$. Fix $u \in G^{(0)}$ and let $\mathcal{H} = \{ x \in G : s(x) = u = r(x) \}$ be the "stability group at u". Then, if G is second countable and has an open range map, $G_u = \{ x \in G : s(x) = u \}$ is a $(G, \mathcal{H})$-equivalence.[a] In particular, G is equivalent to the group $\mathcal{H}$.

[a] Showing that $r|_{G_u}$ is open is non-trivial and requires second countablity.

The Result

The notion of equivalence is a fundamental tool in the theory. For example:

Theorem (Renault's Equivalence Theorem, MRW '87)

If G and H are equivalent and have Haar systems, then $C^*(G)$ and $C^*(H)$ are Morita equivalent.

The importance of equivalence raises the question which is the subject of this talk: "Does equivalence preserve the property of having a Haar system?"

Theorem (W, '15)

Suppose that G is a second countable locally compact groupoid with a Haar system and that H is equivalent to G, then H has a Haar system.

The Imprimitivity Groupoid

1. Suppose that Z is a (G, H)-equivalence.

2. If $(z, w) \in Z *_r Z = \{ (z, w) : r(z) = r(w) \}$ then there is a unique $[z, w]_H \in H$ such that $w = z \cdot [z, w]_H$.

3. This induces a homeomorphism of $G \backslash (Z *_r Z)$ onto H.

4. This is an isomorphism of groupoids provided we equip $G \backslash (Z *_r Z)$ with the induced groupoid structure: $[z, w][w, z'] = [z, z']$ and $[z, w]^{-1} = [w, z]$.

5. If Z is any free and proper G-space with open moment map, then can give $G \backslash (Z *_r Z)$ the groupoid structure as above. Then we call $G \backslash (Z *_r Z)$ the imprimitivity groupoid. (Of course, it's equivalent to G.)

6. Thus, if G has a Haar system and is equivalent to H, then to produce a Haar system on H, it suffices to show that if Z is any free and proper G-space with open moment map, then there is a Haar system on the imprimitivity groupoid $G \backslash (Z *_r Z)$.

- Let $\pi : Y \to X$ be a continuous map. A π-system if a family of Radon measures $\{ \beta^x \}_{x \in X}$ on Y such that $\operatorname{supp} \beta^x \subset \pi^{-1}(x)$ and such that for all $f \in C_c(Y)$,

$$x \mapsto \int_Y f(x) \, d\beta^x(y)$$

  is continuous.
- We say the system is full if $\operatorname{supp} \beta^x = \pi^{-1}(x)$ for all x.
- If Y and X are G-spaces and π is equivariant, then we say β is equivariant if $\gamma \cdot \beta^x = \beta^{\gamma \cdot x}$ provided $s(\gamma) = r(x)$. (Here $\gamma \cdot \beta^x(E) = \beta^x(\gamma^{-1} \cdot E)$.)
- Alternatively, β is equivariant if

$$\int_Y f(\gamma \cdot y) \, d\beta^x(y) = \int_Y f(y) \, d\beta^{\gamma \cdot x}(y)$$

  for all $f \in C_c(Y)$.
- Note that a Haar system is a full equivariant r-system.

First Step

Theorem (KMRW '98)

Suppose that Z is a (G, H)-equivalence. Then the imprimitivity groupoid $H = G\backslash(Z *_r Z)$ has a Haar system if and only if there is full equivariant $r : Z \to G^{(0)}$ system.

Proof of ($\Longleftarrow$).

The unit space of H is (homeomorphic to) the oribt space Z/H. If $\{\alpha^z\}_{z \in Z}$ is an equivariant r-system, then

$$\lambda^{zH}(f) = \int_Z f([z, w]) \, d\alpha^z(w)$$

is a well-defined Haar system on $H = G\backslash(Z *_r Z)$. $\qquad\square$

This reduces our goal to showing that if G has a Haar system and Z is a free and proper G-space, then there is a full equivariant $r : Z \to G^{(0)}$ system.

Theorem (Blanchard, '96)

Let $\pi : Y \to X$ be a continuous surjection. Then π admits a full π-system if and only if π is open.

Comments on the Proof.

Showing that the existence of a full π-system implies π is open is straightforward.

The converse is hard, and follows from a result in Blanchard's Thesis. He shows that if A is a continuous field of C^*-algebras over X, then there is continuous field $x \mapsto \phi_x$ of states on A such that ϕ_x factors through a faithful state on the fibre $A(x)$. The result follows from this. $\qquad\square$

Bruhat Sections or Cut-Off Functions

- Let $\pi : Y \to X$ be a continuous open surjection. A subset $A \subset Y$ is called π-compact if $A \cap \pi^{-1}(K)$ is compact for every $K \subset X$ compact.
- We let $C_{c,\pi}(Y)$ be the vector space of continuous functions on Y with π-compact support.

Lemma (Bruhat Sections)

Let $\pi : Y \to X$ be a continuous open surjection between second countable locally compact Hausdorff spaces. Then there is a $\phi \in C_{c,\pi}^+(Y)$ such that $\pi\big(\{\, y \in Y : \phi(y) > 0 \,\}\big) = X$.

What's left?

We finish off the theorem with the next proposition where we do not assume Z is a free and proper G-space — only that G acts properly on Z!

Proposition

Suppose that G is a second countable locally compact groupoid with a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$, and that Z is a proper G-space. Then there is a full equivariant r_Z-system for the moment map $r_Z : Z \to G^{(0)}$.

Proof.

Blanchard's result implies that there is a full r_Z-system $\{\beta^u\}_{u \in G^{(0)}}$. The idea of the proof is to use the Haar system on G to average this r_Z-system to create an equivariant system.

Note that the properness of the G-action on Z implies that the quotient map $q : Z \to G \backslash Z$ is a continuous and open map onto the Hausdorff space $G \backslash Z$.

Proof of the Key Proposition

Proof Continued.

- Let $\phi \in C_{c,q}^+(Z)$ be such that $q(\{\, z : \phi(z) > 0 \,\}) = G \backslash Z$.
- Define a Radon measure ν^u on Z by

$$\nu^u(f) := \int_G \int_Z f(g \cdot z)\phi(z)\,\beta^{s(g)}(z)\,d\lambda^u(g)$$
$$= \int_G \Psi_\phi(f)(g)\,d\lambda^u(g).$$

- Since λ is a Haar system and since we can show $\Psi_\phi(f) \in C_c(G)$, $u \mapsto \nu^u(f)$ is continuous.
- Easy to see $\operatorname{supp}\nu^u \subset r_Z^{-1}(u)$ and our choice of ϕ guarantees that the support is full.
- All that is left is left-invariance:

Proof.

$$\int_Z f(g' \cdot z) \, d\nu^{s(g')}(z)$$
$$= \int_G \int_Z f(g'g \cdot z)\phi(z) \, d\beta^{s(g)}(z) \, d\lambda^{s(g')}(g)$$

which, since λ is a Haar system, is

$$= \int_G \int_Z f(g \cdot z)\phi(z) \, d\beta^{s(g)}(z) \, d\lambda^{r(g')}(g)$$
$$= \int_Z f(z) \, d\nu^{g' \cdot s(g')}(z). \qquad \square$$

- This completes the proof of the proposition.
- Hence we have proved our our main result as well: the property of having a Haar system is preserved under equivalence.
- The proposition has an interesting corollary for group actions.

Corollary

Suppose that $\mathcal{G}$ is a second countable locally compact group acting properly on a space X. Then X supports at least one $\mathcal{G}$-invariant measure with full support.

Example

Any groupoid which is equivalent to an étale groupoid has a Haar system.

Example (Blanchard)

A proper principle groupoid is a groupoid G for which the action $x \cdot s(x) = r(x)$ is free and proper. If G has open range and source maps, then such a groupoid is equivalent to its (Hausdorff) orbit space $G \backslash G^{(0)}$. Hence proper principle groupoids always have Haar systems provided they have open range maps.

Question

Need a second countable, locally compact, proper principal groupoid have an open range map?

Example (Seda)

Any transitive groupoid G with open range map is equivalent to any of its stability groups. Hence G must have a Haar system.

Question

Must a transitive second countable groupoid have an open range map?

Example

Suppose G is a second countable groupoid with a Haar system acting freely and properly on a space Z. Then the imprimitivity groupoid $G \backslash (Z *_r Z)$ has a Haar system.

The End.

References

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📄 Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *Equivalence and isomorphism for groupoid C*-algebras*, J. Operator Theory **17** (1987), no. 1, 3–22. MR 88h:46123

📄 Dana P. Williams, *Haar systems on equivalent groupoids*, (arXiv.math.OA.1501.04077), 2015.

Lemma

Let G, Z, λ and β be as above.

1. *If $F \in C_c(G \times Z)$, then*

$$\Phi(F)(g, u) := \int_Z F(g, z)\, \beta^u(z)$$

   defines an element of $C_c(G \times G^{(0)})$.

2. *If $f \in C_c(Z)$ and $\phi \in C_{c,q}(Z)$, then*

$$\Psi_\phi(f)(g) = \int_Z f(g \cdot z)\phi(z)\, d\beta^{s(g)}(z)$$

   defines an element of $C_c(G)$.

Proof.

If $F(g, z) = f(z)\phi(z)$, then $\Phi(F)(g, u) = f(g) \int_Z \phi(z) \, d\beta^u(z)$ is clearly continuous. Now approximate to get (1).

Using the properness of the action and the q-compact support of ϕ, it follows that

$$F(g, z) = f(g \cdot z)\phi(z)$$

is in $C_c(G \times Z)$. Then note that

$$\Psi_\phi(f)(g) := \int_Z f(g \cdot z)\phi(z) \, d\beta^{s(g)}(z) = \Phi(F)(g, s(g)).$$

So (2) follows from (1). ☐