# Semigroups, higher rank graphs, and groupoids

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# Introduction

Given a semigroup $P$ (assumed here to be discrete countable), is there a semigroup C*-algebra $C^*(P)$?

G. Murphy has given a natural answer: the universal C*-algebra for isometric representations. However, this C*-algebra is hard to study.

On the other hand, there exist several C*-algebras associated with $P$ which are much more tractable, for example, the left or right reduced C*-algebras $C_l^*(P)$ and $C_r^*(P)$ acting on $\ell^2(P)$ or the Wiener-Hopf C*-algebra $\mathcal{W}(P, Q)$ (assuming, as we shall always do, that $P$ is a subsemigroup of a group $Q$). The reason that these C*-algebras are easier to study is that they can be described as groupoid C*-algebras.

# Introduction (cont'd)

Two classes of semigroups, namely Ore semigroups and quasi-lattice ordered semigroups have been particularly studied. My explanation is that their associated groupoids are easier to grasp.

We shall also see how partial semigroup actions, inverse semigroups, and topological higher rank graphs enter into the groupoid picture.

# The groupoid of germs of an inverse semigroup action

Let us first recall the construction, given by Khoshkam-Skandalis and by Exel, of the groupoid of germs of an inverse semigroup action:
Given a topological space $X$, we denote by $\mathcal{I}(X)$ the pseudogroup of partial homeomorphisms of $X$. An action of an inverse semigroup $\mathcal{S}$ on $X$ is an inverse semigroup morphism $\alpha : \mathcal{S} \to X$.
We then define the groupoid of germs $\mathrm{Germ}(X, \mathcal{S}, \alpha)$ as the quotient of

$$\mathcal{S} * X = \{(s, x) : s \in \mathcal{S}, x \in \mathrm{dom}(\alpha_s)\}$$

by the relation $[s, x] = [t, y]$ iff $x = y$ and there exists $e \in \mathcal{S}^{(0)}$ such that $x \in \alpha_e$ and $se = te$.
It is an étale groupoid.

## The groupoid of an inverse semigroup

If we have just an inverse semigroup $\mathcal{S}$, we first let it act on its set of idempotents $\mathcal{E}$ by $e \mapsto ses^*$. This action is not so interesting but induces an action on the commutative C*-algebra $C^*(\mathcal{E})$, hence on its spectrum $X$. The groupoid $G(\mathcal{S})$ of the inverse semigroup $\mathcal{S}$ is defined as the groupoid of germs of the action of $\mathcal{S}$ on $X$. It satisfies $C^*(\mathcal{S}) = C^*(G(\mathcal{S}))$.

# Semigroup actions

Let us define now a semigroup action:

**Definition**

A left action of a semigroup $P$ on a topological space $X$ is a map

$$T : (n, x) \in P * X \quad \mapsto \quad nx \in X$$

where $P * X$ is an open subset of $P \times X$, such that

1. for all $x \in X$, $(e, x) \in P * X$ and $ex = x$;
2. if $(m, x) \in P * X$, then $(n, mx) \in P * X$ iff $(nm, x) \in P * X$; if this holds, we have $n(mx) = (nm)x$;
3. for all $n \in P$, the map defined by $T_n x = nx$ is a local homeomorphism with domain $U(n) = \{x \in X : (n, x) \in P * X\}$ and range $V(n) = \{nx : (n, x) \in P * X\}$.

# The groupoid of germs of a semigroup action

Assuming that $P \subset Q$, where $Q$ is a group, we now define the groupoid of germs of the action:
We let $\mathcal{S}$ be the sub-inverse semigroup of $Q \times \mathcal{I}(X)$ generated by the elements $(n, T_{n|U})$ where $n \in P$ and $U$ is an open set on which $T_n$ is injective. Then, the restriction $\alpha$ of the second projection of the product $Q \times \mathcal{I}(X)$ si an action of the inverse semigroup $\mathcal{S}$ on $X$. We define the groupoid of germs of $(X, P, T)$ as the groupoid of germs $\mathrm{Germ}(X, \mathcal{S}, \alpha)$ and we denote it by $\mathrm{Germ}(X, P, T)$.

# directed action

Through the groupoid $\mathrm{Germ}(X, P, T)$, we can define the C*-algebra (full or reduced) of the semigroup action $(X, P, T)$. However, very little can be said about it. The following condition will make the groupoid of germs (and its C*-algebra) more tractable.

**Definition**

We say that a semigroup action $(X, P, T)$ is directed if for all pairs $(m, n) \in P \times P$ such that $U(m) \cap U(n) \neq \emptyset$, there exist $(a, b) \in P \times P$ such that $am = bn$ and $U(am) \supset U(m) \cap U(n)$.

## The semi-direct product groupoid

Given a semigroup action $(X, P, T)$ where $P \subset Q$, we define $G(X, P, T)$ as

$$\{(x, m^{-1}n, y) : m, n \in P, x \in U(m), y \in U(n), T_m x = T_n y\}.$$

When $P = \mathbb{N}$ one retrieves the well-known groupoid of an endomorphism studied by V. Deaconu:

$$G(X, T) = \{(x, m - n, y) : m, n \in \mathbb{N}, x, y \in X, T^m x = T^n y\}.$$

In the general case, $G(X, P, T)$ is not necessarily a groupoid (more precisely a subgroupoid of $X \times Q \times X$).

# Identification of the groupoid of germs

However, when the action is directed, it is a groupoid and it is in fact the groupoid of germs defined earlier

**Theorem (R 2015)**

*Assume that the action $(X, P, T)$ is directed. Then*

1. $G(X, P, T)$ *is a locally compact Hausdorff étale groupoid;*
2. $G(X, P, T)$ *is the groupoid of germs of the action.*

This explains a posteriori why both Ore semigroups and quasi-lattice ordered semigroups rather than more general semigroups have been considered.

# The inverse hull of a semigroup

The most obvious actions of a semigroup $P$ are the left and the right actions on $P$ itself by translations. We choose to consider the left action $L$. If $P$ is left cancellative, the left inverse hull $\mathcal{S}_l(P)$ of $P$ is defined as the inverse semigroup generated by the left translations $L(s)$. Then we have the groupoid $G_l(P)$ of the inverse semigroup $\mathcal{S}_l(P)$ and its C*-algebra $C^*_l(P)$.
In this generality, we have little information on $G_l(P)$ hence on $C^*_l(P)$.
By construction, $G_l(P)$ is the groupoid of germs of the action of $P$ on the spectrum $X(P)$ of the commutative C*-algebra $C^*(\mathcal{E})$, where $\mathcal{E}$ is the inf lattice of idempotents, which can be viewed as the subsets of $P$ of the form

$$L(s_1)^{-1}L(t_1)\dots L(s_n)^{-1}L(t_n)(P), \quad s_1, t_1, \dots, s_n, t_n \in P$$

These subsets are called the constructible right ideals by Xin Li.

# The case of Ore and quasi-lattice ordered semigroups

Assume now that $P$ be a left cancellative semigroup. We have two ways to view the left translation of $P$ on itself .

1. As a left action $L(s)t = st$. This action is directed iff $P$ is right reversible, i.e. for all pairs $(m, n) \in P \times P$, $Pm \cap Pn \neq \emptyset$.
2. As a right action of $P$, by defining $L(s)^{-1}(sx) = x$. This action is directed iff $P$ satisfies the Clifford condition: if $mP \cap nP \neq \emptyset$, it is of the form $rP$ for some $r \in P$.

These conditions define respectively the right Ore and the quasi-lattice ordered semigroups.

# The case of Ore and quasi-lattice ordered semigroups (cont'd)

When we pass to the action of $P$ on the spectrum $X(P)$, the directedness of the action is preserved. Therefore, the groupoid of germs can be described as a semi-direct groupoid.

# topological category

Let us turn to topological higher rank graphs. We first need the definition of a topological category. It is just like a topological groupoid but without the existence of an inverse.

## Definition

A topological category is a small category $\Lambda$ with set of objects $\Lambda^{(0)}$, range and source maps $r, s : \Lambda \to \Lambda^{(0)}$, composition map

$$m : \Lambda^{(2)} = \Lambda * \Lambda \to \Lambda,$$

endowed with a topology compatible with its structure.

More precisely, we require that all above maps are continuous and that $s$ is a local homeomorphism.

# topological higher rank graphs

**Definition**

A topological higher-rank graph graded by a semigroup $P$, or $P$-graph for short, is a topological category $\Lambda$ endowed with a map, called the degree map, $d : \Lambda \to P$ which satisfies the following properties

1. for all $m \in P$, $\Lambda^m = d^{-1}(m)$ is open;
2. for all $(\mu, \nu) \in \Lambda^{(2)}$, $d(\mu\nu) = d(\mu)d(\nu)$ and for all $v \in \Lambda^{(0)}$, $d(v) = e$;
3. it has the unique factorization property: for all $m, n \in P$, the composition map $\Lambda^m * \Lambda^n \to \Lambda^{mn}$ is a homeomorphism.

# Examples

1. Graphs are THRG. More accurately, given a graph $(E, V)$, we let
$$\Lambda = \{x_1 x_2 \ldots x_n \quad \text{finite path of the graph}\}$$
Here, $P = \mathbb{N}$ and $d(x_1 x_2 \ldots x_n) = n$. For example, when $V$ has a single element, $\Lambda$ is the set of all finite words with letters in $E$.
2. Topological graphs are THRG. Here $E, V$ are topological spaces, $r, s : E \to V$ are continuous and $s$ is a local homeomorphism.

# from SGA to THRG

**Proposition**

*Let $T : X * P \to X$ be a semigroup action as above. Then $\Lambda = X * P$ has a natural structure of topological higher rank graph.*

It is given by $\Lambda^{(0)} = X$, the range and source maps $r, s : \Lambda \to X$ are respectively $r(x, n) = x$ and $s(x, n) = xn$. The composition is given by

$$(x, m)(xm, n) = (x, mn)$$

The degree map $d : \Lambda \to P$ is simply $d(x, n) = n$.

# from THRG to SGA

I will simply mention that conversely, there is a semigroup action of $P$ on a $P$-graph $\Lambda$, which induces an interesting action on its order compactification $\overline{\Lambda}$, which is the infinite path space defined yesterday.