Coaction Functors and Exact Large Ideals of Fourier-Stieltjes Algebras

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Thank You!

References

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3. ____, *Exotic coactions*, `arXiv:1305.5489`
4. Baum-Guentner-Willett, *Expanders, exact crossed products, and the Baum-Connes conjecture*, `arXiv:1311.2343`
5. Buss-Echterhoff-Willett, *Exotic crossed products and the Baum-Connes conjecture*, `arXiv:1409.4332`
6. K-Landstad-Quigg, *Coaction functors*, `arXiv:1505.03487`
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Any Questions?

- Does every crossed product functor come from a coaction functor?
- Does every coaction functor come from some large ideal? (No.)
- Does every exact one come from some exact large ideal?
- Which large ideals are exact?
- If G is exact and $2 \leq p < \infty$, is $E_p = B(G) \cap L^p(G)$ exact?

Conclusion

For any large ideal E of $B(G)$, the assignment

$$\tau_E\colon (A,\delta) \mapsto (A^E, \delta^E)$$

is functorial on $\widehat{G}$-C^*.

There are surjective natural transformations

$$(A^m, \delta^m) \to (A^E, \delta^E) \to (A^n, \delta^n).$$

(A^E, δ^E) is Morita equivalent to (B^E, ϵ^E) whenever (A, δ) is Morita equivalent to (B, ϵ).

τ_E preserves short exact sequences (E is exact) if and only if

$$\{a \in A \mid E \cdot a \subseteq I\} \subseteq I + \{a \in A \mid E \cdot a = 0\}$$

for all (A, δ) and strongly invariant $I \subseteq A$.

If E and F are both exact, then $E \cap F$ is also exact.

Large Ideals of $B(G)$

An ideal E of $B(G) = C^*(G)^* \subseteq C_b(G)$ is *large* if E is weak-$*$ closed, contains $B_r(G) = C^*_r(G)^*$, and is invariant under the natural action(s) of G on $B(G)$:

$$(x \cdot f \cdot y)(z) = f(xzy)$$

$$\begin{aligned} \text{large ideal } E \text{ of } B(G) &\mapsto \text{small ideal } {}^{\perp}E \text{ of } C^*(G) \\ &\mapsto \text{exotic group } C^*\text{-algebra} \\ &\quad\ C^*_E(G) = C^*(G)/{}^{\perp}E \\ &\quad\ \text{with coaction } \delta^E_G \text{ of } G \end{aligned}$$

Large Ideals of $B(G)$

Given a coaction (A, δ) of G:

$$\begin{aligned} \text{large ideal } E &\mapsto \text{small } \delta\text{-invariant ideal} \\ &\qquad A_E = \{a \in A \mid E \cdot a = 0\} \\ &\mapsto \text{an exotic coaction } \delta^E \\ &\qquad \text{of } G \text{ on } A^E = A/A_E \end{aligned}$$

- $\tau_{B_r(G)}$ is normalization $(A, \delta) \mapsto (A^n, \delta^n)$.
- $\tau_{B(G)}$ is the identity functor $(A, \delta) \mapsto (A, \delta)$.
- Maximalization $(A, \delta) \mapsto (A^m, \delta^m)$ is not τ_E for any E.

The Baum-Connes Conjecture(s)

For any second-countable locally compact group G, and any G-C^*-algebra A, the (reduced) assembly map

$$\mu_{\mathrm{red}}\colon K_*^{top}(G;A) \to K_*(A \rtimes_{\mathrm{red}} G)$$

is an isomorphism.

"Counterexamples. . . are closely connected to failures of *exactness*"

Maybe the *maximal* assembly map

$$\mu_{\max}\colon K_*^{top}(G;A) \to K_*(A \rtimes_{\max} G)$$

is an isomorphism.

"There are well-known Property (T) obstructions. . . "

Crossed Product Functors

"Study crossed products that combine the good properties of the maximal and reduced crossed products."

A *crossed product* is a functor

$$A \mapsto A \rtimes_\tau G$$

from G-C^* to C^*, together with natural transformations

$$A \rtimes_{\max} G \to A \rtimes_\tau G \to A \rtimes_{\mathrm{red}} G$$

restricting to the identity map on the dense subalgebra(s) $A \rtimes_{\mathrm{alg}} G$.

Each has a τ*-assembly map*

$$\mu_\tau \colon K_*^{top}(G; A) \to K_*(A \rtimes_{\max} G) \to K_*(A \rtimes_\tau G).$$

Crossed Product Functors

τ is *exact* if

$$0 \to I \rtimes_\tau G \to A \rtimes_\tau G \to B \rtimes_\tau G \to 0$$

is short exact in C^* whenever $0 \to I \to A \to B \to 0$ is short exact in G-C^*.

τ is *Morita compatible* if

$$A \rtimes_\tau G \sim B \rtimes_\tau G$$

whenever $A \sim B$ equivariantly (roughly speaking).

$\sigma \leq \tau$ if the natural transformations factor this way:

$$A \rtimes_{\max} G \to A \rtimes_\tau G \to A \rtimes_\sigma G \to A \rtimes_{\mathrm{red}} G$$

Crossed Product Functors

$\max$ is Morita compatible and exact, and $\mathrm{red} \leq \max$.

red is Morita compatible, but not (in general) exact.

Theorem. For any second countable locally compact group G, there exists a unique minimal exact and Morita compatible crossed product $\mathcal{E}$.

Conjecture: For any G-C^*-algebra A, the $\mathcal{E}$-assembly map

$$\mu_{\mathcal{E}}\colon K_*^{top}(G; A) \to K_*(A \rtimes_{\mathcal{E}} G)$$

is an isomorphism.

Coaction Functors

A *coaction functor* is a functor

$$(A,\delta) \mapsto (A^\tau,\delta^\tau)$$

on $\widehat{G}$-C^*, together with surjective natural transformations

$$(A^m,\delta^m) \to (A^\tau,\delta^\tau) \to (A^n,\delta^n),$$

where $(A,\delta) \mapsto (A^m,\delta^m)$ is the *maximalization functor*, and $(A,\delta) \mapsto (A^n,\delta^n)$ is the *normalization functor*.

By the way,

$$A^m \rtimes_{\delta^m} G \rtimes_{\mathrm{max}} G \cong A^m \otimes \mathcal{K},$$

and

$$A^n \rtimes_{\delta^n} G \rtimes_{\mathrm{red}} G \cong A^n \otimes \mathcal{K}.$$

Coaction Functors

τ is *exact* if

$$0 \to (I^\tau, \gamma^\tau) \to (A^\tau, \delta^\tau) \to (B^\tau, \epsilon^\tau) \to 0$$

is short exact in $\widehat{G}$-C^* whenever $0 \to (I, \gamma) \to (A, \delta) \to (B, \epsilon) \to 0$ is short exact.

τ is *Morita compatible* if $(A^m, \delta^m) \sim (B^m, \epsilon^m)$ descends to

$$(A^\tau, \delta^\tau) \sim (B^\tau, \epsilon^\tau)$$

whenever $(A, \delta) \sim (B, \epsilon)$ (roughly speaking).

$\sigma \leq \tau$ if the natural transformations factor this way:

$$(A^m, \delta^m) \to (A^\tau, \delta^\tau) \to (A^\sigma, \delta^\sigma) \to (A^n, \delta^n).$$

Coaction Functors

Maximalization is Morita compatible and exact, and dominates normalization.

Normalization is Morita compatible, but not (in general) exact.

Theorem. There exists a unique minimal exact and Morita compatible coaction functor.

Coaction Functors

Theorem. For any coaction functor τ, the composition σ defined by

$$\begin{aligned}\sigma\colon (A,\alpha) &\mapsto (A \rtimes_{\max} G, \hat{\alpha}) \\ &\mapsto ((A \rtimes_{\max} G)^\tau, (\hat{\alpha})^\tau) \\ &\mapsto (A \rtimes_{\max} G)^\tau\end{aligned}$$

(with the associated natural transformations) is a crossed product functor.

- If τ is exact, then σ will be exact;
- If τ is Morita compatible, then σ will be Morita compatible;
- If $\tau \leq \tau'$, then $\sigma \leq \sigma'$.
- If $\tau = \tau_E$, we write $(A \rtimes_{\max} G)^\tau = A \rtimes_E G$.

Abstract

We use functors on a category of coactions to understand the crossed-product functors that have been recently introduced by Baum, Guentner and Willett in relation to the Baum-Connes conjecture. The most important coaction functors (to us) are the ones induced by "large" ideals of a Fourier-Stieltjes algebra $B(G)$. The intersection of two large ideals of $B(G)$ for which the associated coaction functors (and hence the associated crossed-product functors) are exact, is also exact.

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